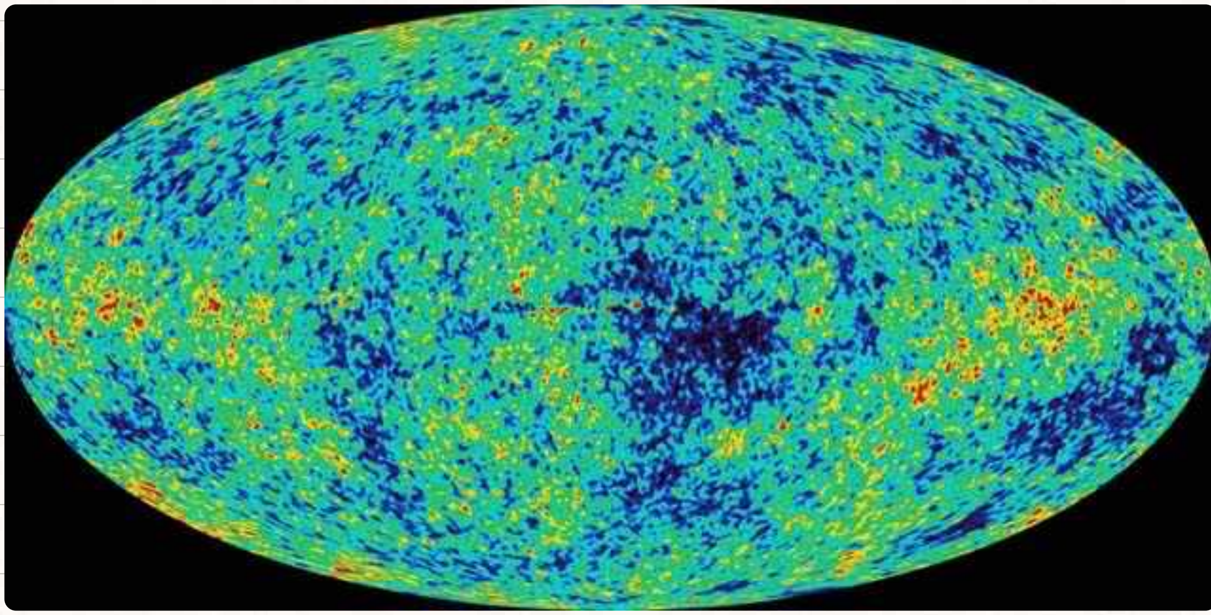
Singularity Theorems in Lorentzian Geometry



http://cosmology.berkeley.edu/Education/CosmologyEssays/The_Cosmic_Microwave_Background.html

Goal: Formulate general conditions on spacetime which imply the existence of singularities.

What is a Singularity?

Symmetry-examples: $\rho \to \infty$, $R \to \infty$, incomplete geodesics

Def. 1 (Singular spacetime)
We call a spacetime (M, g) singular if there exist incomplete causal geodesics.

Or Lorentzian version of Myers's Theorem:

Theorem (Myer)

Let (M, g) be a complete smooth Riemannian manifold of dimension n.

If $k > 0$ and $Ric(V, V) \geq (n-1) k\, g(V, V)$ $\forall\, V \in TM$

$$\Rightarrow \begin{cases} \cdot\ M \text{ compact} \\ \cdot\ diam(M) \leq \pi / \sqrt{K} \\ \cdot\ \pi(M) \text{ finite} \end{cases}$$

(I) The Arena

(•) M smooth manifold (Hausdorff, second countable)

(•) g Lorentzian metric : Symmetric non-degenerate $(0,2)_k$-tensor field of index 1.

Regularity of g?
- in classical Singularity Theorems $k \geq 2 \rightarrow$ normal neighbourhoods
- new result by Melanie Graf (2020) : $k = 1$

Why do we want low regularity?

- when describing stars we need to match spacetimes (exterior & interior)
  $\rightarrow$ classical solution of the Einstein-Field equations:
  Oppenheim-Snyder solution $\Rightarrow g \in C^{1,1}$
- also when describing shock waves $\Rightarrow g \in C^{1,1}$
- Philosophical : If the theorems would break at low regularity their consequences could be circumneviyated by a spacetime which is of low regularity.

If $g \in C^1$: • Ric well defined as distribution
• $\exists$ of geodesics, even though not unique (Peano)

The Arena now gives us a language in which we can formulate <u>causality</u>

<u>Remark</u>: Class of timelike curves $\Rightarrow$ Topology of spacetime

<u>(II) Causality in (T_pM, g_p)</u>

- inspired by SRT $\begin{pmatrix} T_pM \equiv \text{lin approx of } M \\ SRT \equiv \text{lin. approx. of GR} \end{pmatrix}$

<u>Def 2.1</u>: $v \in T_pM$: $\begin{cases} \text{causal} : g_p(v,v) \leq 0, v \neq 0 \\ \text{timelike} : g_p(v,v) < 0 \\ \text{null} : g_p(v,v) = 0, v \neq 0 \\ \text{spacelike} : g_p(v,v) > 0 \text{ or } v = 0 \end{cases}$

- $\Pi_p := \{ v \in T_pM \mid g_p(v,v) < 0 \}$ (timelike vectors)

<u>Def 2.2</u> : $v, w \in \Pi_p$: $v \sim w :\Longleftrightarrow g_p(v,w) < 0$

<u>Remark</u>

(•) $\sim$ is an equiv. relation

$C(u) := \{ v \in \Pi_p : g_p(u,v) < 0 \}$ $u \in \Pi_p$

$\hookrightarrow \Pi_p = \underset{\substack{\text{III} \\ \text{future direction}}}{C(u)} \sqcup \underset{\substack{\text{III} \\ \text{past direction}}}{C(-u)}$

<u>In SRT</u> : $(T_pM, g_p) \hat{=} (M, g) = (\mathbb{R}^4, \eta)$

$\hookrightarrow$ $\sim$ defines at every point the causal future and past

$M \neq \mathbb{R}^4$?

Def. 2.2: We call a Lorentzian manifold (M, g) time-orientable if there exists a global timelike vector field $u \in \mathfrak{X}(M)$

($\Leftrightarrow$ we can define the causal future and past direction at every point in a smooth way)

Def 2.3

We call $\alpha : I \to M$ a locally lipschitz continuous curve :

- timelike if almost everywhere : $\dot{\alpha}(t) \in T_{\alpha(t)} M$ timelike
- causal, null, spacelike ...
- future directed : $g(\dot{\alpha}, u_{\alpha(t)}) < 0$

Def (2.4) $A \subseteq M$

$I^+(A) := \{ q \in M \mid \exists \alpha$ timelike curve from A to $q \}$

$J^+(A) := \{ q \in M \mid \exists \alpha$ causal curve from A to $q \}$

$E^+(A) := J^+(A) - I^+(A)$

In SRT: $I^+(p) = C(U_p)$, $J^+(p) = \overline{C(U_p)}$, $E^+(p) = \partial C(U_p)$

$\Big\downarrow$ $\exp_p$, 'partial' isometry

<u>Theorem (2.7)</u> $\quad g \in C^2$

Let $\mathcal{V}$ be a normal neighbourhood of $p \in M$.
<u>Then</u>:

$$y = \exp_p(X) \in \begin{cases} I^+(p, \mathcal{V}) \iff x \in C(U_p) \\ J^+(p, \mathcal{V}) \iff x \in \overline{C(U_p)} \\ E^+(p, \mathcal{V}) \iff x \in \partial C(U_p) \end{cases}$$

<u>Idea</u>: As in Riemannian geometry one proves the Gauß Lemma:

$$g_p(X, w_x) = g_{\exp_p(x)}(d\exp_p(x), d\exp_p(w_x))$$

Intuitive result, tedious prove ...

An unintuitive result, simpler prove:

Theorem (2.2) (Twin-paradox)

Let $\mathcal{V}$ be a normal neighbourhood of $p \in M$. For $q \in \mathcal{I}^+(q, \mathcal{V})$ the unique ($\sim$ parametrization) longest curve from p to q is given by the radial geodesic:
$\exp_p^{-1}(q)t$ (in coord.)

Remark (2.3)

· the 'length' is measured by the proper time:

$$L_g(\alpha) := \int_I |\dot{\alpha}|_g \, dt.$$

· We define the time separation of $p, q \in M$:

$$\tau(p,q) := \sup \{ L_g(\alpha) \mid \alpha \text{ causal from } p \text{ to } q \}$$

$\hookrightarrow$ $\exists$ of maximizing curves? $\rightarrow$ study limit curves

→ causal curves as 'atoms' of causality

(III) Limit curves

Theorem 3.34. *(limit curve theorem I) (cf. [14] Theorem 1.5 p.5 and [11] Prop. 2.6.1/2.6.7 p.34)*
Let $(\alpha_n)_n$ be a sequence of LLC-causal curves, such that $\alpha_n(0) \to p \in M$. If furthermore one of the following is given:

1. all α_n are proportional to h-arclength parametrized, are defined on the interval $[0,1]$ and have bounded h-arclengths from both sides: $C' > L_h(\alpha_n) > C > 0$.[5]
2. all α_n are inextendible

then there exists a causal curve α starting at p such that there is a subsequence $(\alpha_{n_k})_k$ which converges to α uniformly on compact sets.
In the first case this implies uniform convergence on $[0,1]$. If the second condition is fulfilled instead, it follows that α is inextendible too.

- h complete Riemannian background metric
- α_n proportional to h-arclength on $[0,1]$
  → $|\dot{\alpha}_n|_h = L_h(\alpha_n)$ a.e.

- $\alpha_n : [0,b) \to M$ (future) inextendible if $\lim_{t \to b} \alpha_n(t)$ does not exist

Proof: Arzelà Ascoli

Idea of our Singularity Theorem:

(1) Find a causality condition which implies the existence of maximal geodesics

(2) Find an initial and Energy/Curvature-condition which implies the failure of maximality after a finite proper time

(3) If there would exist timelike curves of arbitrary long proper time there would exist maximal geodesics of arbitrary long proper time $\lightning$

(IV) Global hyperbolicity

Def (4.1)

A spacetime $(\mathcal{M}, g)$ is called globally hyperbolic if:

(i) $(\mathcal{M}, g)$ is non-totally imprisoning
: $\nexists$ no future / past inextendible causal curve contained in a compact set

(ii) $\forall p, q \in M$: $J(p, q) := J^+(p) \cap J^-(q)$ is compact.

Limit curve theorems inspire the following definition:

$$\cdot \; \hat{C}(p, q) := \overline{C_h(p, q)}^{\tau_{C^0}}$$

$$\cdot \; C_h(p, q) := \left\{ \alpha : [0,1] \to M \;\middle|\; \begin{array}{l} \alpha \text{ causal}, \\ \alpha(0) = p, \; \alpha(1) = q \\ |\dot{\alpha}|_h = \text{const} \end{array} \right\}$$

<u>Remark</u>: One can show that condition (ii) in Def (4.1) can instead be formulated as $\hat{C}(p, q)$ being compact $\forall p, q \in M$

Lemma (4.2)

If (M, g) is globally hyperbolic and $q \in J^+(p)$
$\Rightarrow \exists \gamma$ causal geodesic from p to q such that
$L_g(\gamma) = \tau(p, q)$.

Further characterization of global hyperbolicity:

Def (4.3)
We call $S \subseteq M$ a Cauchy hypersurface if it is met exactly once by every inextendible causal curve.

Remark Let $S \subseteq M$ be a Cauchy hypersurface.

- S is a closed acausal topological hypersurface
- $M = I^-(S) \cup S \cup I^+(S)$ (disjoint)
- M is globally hyperbolic and $M \cong \mathbb{R} \times S$

This new characterization motivates us to examine maximal geodesics starting from a hypersurface.

$\downarrow$

(V) Calculus of Variations

→ study extremal properties of L_g
→ find conditions which prevent maximality

Notation

- $\gamma : [0,b] \to M \quad \in \mathcal{C}^2_{pc} \quad ; \ |\dot\gamma|_g = 1$, timelike
- $\Gamma : [0,b] \times (-\varepsilon, \varepsilon) \to M \quad \in \mathcal{C}^2_{pc}$,
  $\Gamma(t,0) = \gamma(t)$
  $V(t) := \partial_s \Gamma(t,s)|_{s=0}$ ('Variation vectorfield')
  → Γ <u>Variation of γ</u>

- $P \subseteq M$ spacelike hyper surface
- Γ is a (P,g)- Variation if
  $\Gamma(b,s) = \gamma(b) = q, \quad \Gamma(0,s) \in P$

∘ First variation

$$d_\gamma L(V) \equiv \frac{d}{ds}\Big|_{s=0} L_g(\Gamma(-,s))$$

$$= \int_0^b g(\ddot{\gamma}, V)\, dt$$

$$\sum_{i=1}^{k} g(\Delta\dot{\gamma}, V)(t_i)$$

$$- \; g(\dot{\gamma}, V)\Big|_0^b$$

$\Longrightarrow$ A $\mathcal{C}^2_{pc}$-curve γ of constant speed $|\dot{\gamma}|_g = c > 0$ fulfills $d_\gamma L(V) = 0$ for every $(P, \gamma(b))$-Variation if and only if γ is a geodesic normal to P.

From now on let γ be a <u>geodesic</u> which starts orthogonal to P a spacelike hypersurface

○ Second Variation (Synge's Formula)

- $I_2^{\perp}(\overset{\perp}{V}_1, \overset{\perp}{V}_2) = \left.\dfrac{d^2}{ds^2}\right|_{s=0} L_g(\Gamma(-, s))$

bilinear form

$$= -\int_0^b \{ g(\overset{\perp}{V}', \overset{\perp}{V}') - R(\overset{\perp}{V}, \dot{\gamma}, \overset{\perp}{V}, \dot{\gamma}) \} dt + g(\dot{\gamma}(0), \underset{\sim}{\mathrm{II}}(V(0), V(0)))$$

- $\mathrm{II}(X, Y) = \underline{\perp}\, {}^{i}\nabla_X\, di\,(Y)$ (second fundamental form)

- $i : (P, g') \hookrightarrow (M, g)$ isometric Immersion

- $\underline{\perp} : {}^{i}TM \to N_i$

Def (5.1)

- We call V a P-Jacobi-field if it is a variation vector field through P-normal geodesics

Def (5.2) (Focal point)

We call $\gamma(b)$ a focal point of P along γ if there is a P-Jacobi field V on γ such that $V(b) = 0$.

Theorem (5.4) (O'Neil Theorem 34)

Let γ be a geodesic such that:
$\gamma(0) \in P$, $\dot{\gamma}(0) \perp P$. Then :

(1) If there are no focal points of P along γ, then $I_\gamma^\perp$ is negative definite

(2) If $q := \gamma(b)$ is a focal point along γ then $I_\gamma^\perp$ is semi-definite, but not definite

(3) If $\gamma(a)$ is a focal point such that $0 < a < b$ then $I_\gamma^\perp$ is not semi-definite

Important conclusion:

If we can somehow prove the existence of focal points along a (normal) geodesic we would have also proven that γ is not maximal

$\rightarrow$ Exactly what we wanted to show for our Singularity Theorem

Def. 5.5 (convergence)

We define the convergence of P as the real valued function on the normal bundle NP:

- $K(z) := g(z, H_p) = \frac{1}{n-1} \operatorname{tr}(S_z)$

where $H_p := \frac{1}{n-1} \sum_{i=1}^{n-1} II(e_i, e_i)$; $z \in T_p(P)^{\perp}$

is the <u>mean curvature field</u>

To further analyze the existence of focal points we need to use Synge's Formula

$$\circledast \; I_\gamma^{\perp}(V^{\perp}, V^{\perp}) = -\frac{1}{c} \int_0^b \{ g(V^{\perp\prime}, V^{\perp\prime}) - R(V^{\perp}, \dot{\gamma}, V^{\perp}, \dot{\gamma}) \} dt + \frac{1}{c}\, g(\dot{\gamma}(0), II(v(0), v(0)))$$

↳ Now let $f : [0, b] \to M$ be a piecewise smooth function such that $f(0) = 1$, $f(b) = 0$.

$\Rightarrow$ for every $e_i \in T_{\gamma(0)} P$, (e_i): ONB ; we can parallel translate e_i along γ to get a variation vector field:

$V^i(t) := f(t)\, e_i(t)$

$$I_\gamma(V^i, V^i) = -\frac{1}{c}\int_0^b \{(f')^2 - f^2 R(e_i, \dot\gamma, e_i, \dot\gamma)\}dt$$

$$+ \frac{1}{c}\, g(\dot\gamma(0), II(e_i, e_i))$$

$$\sum_{i=1}^{n-1} I_\gamma(V^i, V^i) = -\frac{1}{c}\int_0^b \{(n-1)(f')^2 - f^2 Ric(\dot\gamma, \dot\gamma)\}dt$$

$$+ \frac{1}{c}(n-1)\, k(\dot\gamma(0))$$

$$\equiv \mathcal{Y}[f]$$

$\Longrightarrow$ If we can find an f such that $\mathcal{Y}[f] \geq 0 \Rightarrow$ <u>$\exists$ of a focal point</u>

$\rightarrow$ just <u>guess</u>!

<u>Example (5.6 / O'Neil Prop. 37)</u>

$$\left.\begin{array}{l}(1)\ k(\dot\gamma(0)) > 0 \\ (2)\ Ric(\dot\gamma, \dot\gamma) \geq 0\end{array}\right] \xrightarrow[\substack{f(t) := 1 - k/t \\ b \geq 1/k}]{} \mathcal{Y}[f] \geq 0$$

<u>Variation</u>:
- $k(\dot\gamma(0)) \geq \beta > 0$;
- $Ric(\dot\gamma, \dot\gamma) \geq -\delta$; $0 \leq \delta \leq \frac{3\beta}{b}(1-c)$; $0 < c \leq 1$

$\Rightarrow$ local point if $b \geq 1/c\beta$

Excursion : Energy conditions

Since the Einstein-field equations

$$G_{\mu\nu} = Ric_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}$$

connect curvature to the energy-momentum tensor we can formulate curvature conditions as $Ric(X,X) \geq 0$ as Energy conditions:

$$\boxed{T(X,X) \geq \frac{tr(T)}{n-1} \, g(v,v)}$$

This condition is fulfilled by most of the reasonable classical matter-models in our universe.

Though if one incooperates Quantum-mechanical effects this so called strong energy condition is most certainly not fulfilled.

A concrete example is given by the Klein-Gordon-field (scalar field) described by the Klein-Gordon-equation

$$(\Box_g + m^2 + \xi R)\phi = 0$$

(non-minimally coupled)

with Energy-momentum Tensor:

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) + \frac{1}{2} g_{\mu\nu} (m^2 \phi^2 - (\nabla\phi)^2)$$
$$+ \xi (g_{\mu\nu} \Box_g - \nabla_\mu \nabla_\nu - G_{\mu\nu}) \phi^2$$

$\downarrow$

$\dots$ $\mathrm{Ric}(X, X) \geq 0$

$\downarrow$

<u>Quantum-Energy inequalities</u> (C. J. Fewster / E. A. Kontou)
(inspired, still classical though...)

<u>Worldline inequality</u>:

If $\cdot\ |\phi| \leq \phi_{max} \leq (8\pi\xi)^{-1/2}$
$\cdot\ |\nabla_i \phi| \leq \phi'_{max} < \infty$
$\cdot\ \gamma : I \to M$ causal geodesic

$$\boxed{\begin{gathered}\int_\gamma \mathrm{Ric}(\dot\gamma, \dot\gamma) f^2 \, dt \geq Q \|f\|^2_{L^2} + \tilde{Q} \|f'\|^2_{L^2} \\ \cdot\ \forall f \in W^1_0(I) \, ; \, Q < 0 \, , \, \tilde{Q} \leq 0\end{gathered}}$$

<u>Is it still possible to predict focal points?</u>

... yes under some further conditions
$\mathrm{Ric}(\dot\gamma, \dot\gamma) \geq 0$ initially, γ extendable to the past, ...

Back to the classical theorem.

(VI) Hawkings singularity Theorem

We only need one further preparatory Lemma:

Lemma 6.1 (O'Neil (44))

- Let $S \subseteq M$ be a Cauchy hypersurface and $q \in M$.

$\Rightarrow$ There exists a geodesic from S to q of length $\tau(S, q)$.

Remark: $\tau(S,q) := \sup_{s \in S} \tau(s, q)$

Proof (6.1) (sketch)

- $\tau : M \times M \to \mathbb{R}$ is continuous if M is globally hyperbolic (lower semi-continuity always from twin paradox)

- $J^-(q) \cap S$ is compact

$\Rightarrow \exists p \in S: \quad \tau(S,q) = \tau(p,q)$

$\overset{\text{Lemma}}{\underset{4.3}{\Rightarrow}}$ $\exists$ causal geodesic $\gamma : L_g(\gamma) = \tau(p,q) = \tau(S,q)$

(γ then has to be normal)

Theorem (6.2) (Hawking)

Let (M,g) be time-orientable Lorentzian manifold with $g \in C^2$. If the strong energy condition: $Ric(X,X) \geq 0 \ \forall X \in \mathfrak{X}(M)$ holds and there exists a spacelike Cauchy-hypersurface S such that $K(n) \geq \beta > 0$ on S with n the unit normal vector, then:

$$\tau(S,q) \leq 1/\beta \quad \forall q \in I^+(S)$$

, in particular (M,g) is singular.

Proof (6.2) (/ O'Neil / Thm. 55a)

- Let $q \in I^+(S)$. By Lemma (6.1) there exists a (timelike) normal

geodesic γ from S to such that
$L_g(\gamma) = \tau(S, y)$.

- Since γ is maximizing it cannot have any focal points.

- The example in (IV) showed that
$Ric(X, X) \geq 0$, $k(n) \geq \beta > 0$
implies the existence of a focal point after $t \geq 1/\beta|\dot{\gamma}|$

$\Rightarrow \quad L_g(\gamma) = \tau(S, y) \leq 1/\beta$

- Since $y \in I^+(S)$ was arbitrary the claim follows. //

Whats about the

Regularity question?

(VII) The question of regularity

What remains true if the metric is only of C^1 regularity?

A lot! Most importantly:

<u>Theorem 7.1</u> (Melanie Graf)

Let (M, g) be globally hyperbolic and $g \in C^1$.

- Then for any $q \in J^+(p) - p$ there exists at least one maximizing geodesic γ from p to q.

- Moreover γ can be obtained as the C^1-limit of a sequence of maximizing $\check{g}_{\varepsilon_n}$-causal geodesics γ_{ε_n} for $\varepsilon_n \to 0$.

All $\check{g}_\varepsilon$ can be chosen with smaller lightcones than g:

if $\check{g}_{\varepsilon_n}(X,X) \leq 0 \Rightarrow g(X,X) < 0$.

Problem :

$$\mathrm{Ric}_{ij} = \partial_m \Gamma^m_{ij} - \partial_j \Gamma^m_{ij} + \Gamma^m_{ij} \Gamma^k_{km} - \Gamma^m_{ik} \Gamma^k_{jm}$$

with $\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$

$\Rightarrow$: $g \in C^1 \Rightarrow \Gamma^k_{ij} \in C^0 \Rightarrow \mathrm{Ric}_{ij}$ i.g. not well defined

But! We can define $\partial_m \Gamma^k_{ij}$ as a Distribution :

$$\rightarrow \langle \partial_m \Gamma^k_{ij}, \phi \rangle := -\int_\Omega \Gamma^k_{ij} \, \partial_m \phi \, d\Omega$$

$\Omega \in \mathbb{R}^n$

On a Manifold?

$\downarrow$

11 Densities on vector spaces

- W (real) Vector space of dim n
- $\Lambda^n W$ - n fold antisymmetrized tensor product of W

Def 7.2 (q-density)

- For all $q \in \mathbb{R}$ we call: $\mu : \Lambda^n W_{\setminus \{0\}} \to \mathbb{R}$ a <u>q-density</u> if for all $0 \neq s \in \mathbb{R}$ and $0 \neq \omega \in \Lambda^n W$:

$$\mu(s\omega) = |s|^q \mu(\omega)$$

$\Rightarrow$ $\mathrm{Vol}^q(W)$ real one-dimensional vectorspace

(*) (v^i), (w^i) basis of W
$A = (a^{ij})$ matrix of basis change
$v^i = \sum_j a^{ij} w^j$

$$\Rightarrow \mu(v_1 \wedge \dots \wedge v_n) = |\det(A)|^q \mu(w_1 \wedge \dots \wedge w_n)$$

2) Densities on manifolds

<u>Def</u> (7.3) (q-density)

Let $(V_\alpha, \psi_\alpha)_\alpha$ be an Atlas for M.
We call the one-dimensional vector bundle (line bundle) given by the cocycle of transition functions:

$$t_{\alpha\beta} : \psi_\beta(V_\alpha \cap V_\beta) \to \mathbb{R}\setminus\{0\} = GL(1,\mathbb{R})$$
$$t_{\alpha\beta}(y) := |\det D(\psi_\alpha \circ \psi_\beta^{-1})(y)|^{-q}$$
$$= |\det D(\psi_\beta \circ \psi_\alpha^{-1})(\psi_{\alpha\beta}(y))|^{q}$$

Denoted as $Vol^q(M)$,

In the following we will always consider $q = 1$.

<u>Concrete description</u>:

- $Vol(M) = \bigcup_{p \in M} \{p\} \times Vol(T_pM)$

- (V_α, ψ_α) chart: $\exists$ unique density $|dx^1 \wedge \dots \wedge dx^n|(\partial x_1 \wedge \dots \wedge \partial x_n)|_p = 1$

$\downarrow$ Trivializations given as:

$\chi_\alpha(p, \mu_p) = (p, \mu_p(\partial x_1 \wedge \dots \wedge \partial x_n(p)))$

$\hookrightarrow \Phi_\alpha(p, \mu_p) := (\psi_\alpha(p), \mu_p(\partial x_1 \wedge \dots \wedge \partial x_n(p)))$

- local expression of a $\mathcal{C}^k$-section: $\mu \in \Gamma^k(M, \mathrm{Vol}(M))$ is given by:

$$\mu_\alpha = (\Phi_\alpha)_* (\mu|_{V_\alpha}) = \Phi_\alpha \circ \mu|_{V_\alpha} \circ \psi_\alpha^{-1}$$

$$\rightarrow \quad \mu|_{V_\alpha} = (\mu_\alpha \circ \psi_\alpha) \, |dx^1 \wedge \dots \wedge dx^n|$$

- Transformation of local expressions:

$$\mu_\alpha(x) = |\det(D(\psi_\beta \circ \psi_\alpha^{-1}))|(x) \, \mu_\beta(\psi_\beta \circ \psi_\alpha^{-1}(x))$$

exactly the transformation rule which is needed to use the Trafo-formula for the Lebesgue measure!

<u>Def (7.4)</u> (Integral on manifolds)

Let $\mu \in \Gamma_c^0(M, \mathrm{Vol}(M))$, $(V_\alpha, \psi_\alpha)_\alpha$ an Atlas with $\overline{V_\alpha}$ compact and $(\xi_\alpha)_\alpha$ a partition of unity subordinate to $(V_\alpha)_\alpha$

$\downarrow$

$$\int_M \mu = \sum_\alpha \int_{V_\alpha} \xi_\alpha \mu := \sum_\alpha \int_{\psi_\alpha(V_\alpha)} \xi_\alpha(\psi_\alpha^{-1}(x)) \mu_\alpha(x) \, dx$$

We aim to define distributions on

manifolds as the topological dual of $\Gamma_c(M, Vol(M))$.

$\hookrightarrow$ need to define a topology on $\Gamma_c(M, Vol(M))$ ... not here ...

<u>Def</u> (7.5) (Distributions)

- We define the space of distributions on M as $D'(M) := \Gamma_c(M, Vol(M))'$

- We define the space of (r,s)-tensor distributions as : $D'\mathcal{T}^r_s(M) := \Gamma_c(M, T^s_r M \otimes Vol(M))$

$$\hat{=} \Gamma_c(M, Vol(M)) \otimes_{C^\infty} \mathcal{T}^r_s(M)$$
$$= D'(M) \otimes_{C^\infty} \mathcal{T}^r_s(M)$$

(• $D'\mathcal{T}^r_s(M)$ is a fine sheaf of C^∞-Modules)

$\downarrow$

Let $T \in D'\mathcal{T}^r(M)$; (V_α, ψ_α) Atlas

(i) $T|_{V_\alpha} = ({}^\alpha T)^{i_1 \cdots i_r}_{j_1 \cdots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$
$\in D'(V_\alpha)$

(ii) $u \in D'(V_\alpha)$: $\langle (\psi_\alpha)_* u, \varphi \rangle := \langle u, (\Phi_\alpha)^* \varphi \rangle$

$\rightarrow ({}^\alpha \tilde{T})^{i_1 \cdots i_r}_{j_1 \cdots j_s} := (\psi_\alpha)_* \left(({}^\alpha T)^{i_1 \cdots i_r}_{j_1 \cdots j_s} \right) \in D'(\psi_\alpha(V_\alpha))$

(iii) $\{ ((^{\alpha}\hat{T})^{i_1 \cdots i_r}_{j_1 \cdots j_s}) \in \mathcal{D}'(x_\alpha(V_\alpha)) \}_{\alpha \in A}$

$\Longrightarrow \exists\, T \in \mathcal{D}'\mathcal{T}^r_s(M)$ such that

$$(x_\alpha)_* ((^{\alpha}T)^{i_1 \cdots i_r}_{j_1 \cdots j_s}) = (^{\alpha}\hat{T})^{i_1 \cdots i_r}_{j_1 \cdots j_s}$$

//

<u>Def (7.6)</u> :

• Let $u \in \mathcal{D}'(M)$.

We say u is non-negative : $u \geq 0$

if $\langle u, \mu \rangle \geq 0 \quad \forall \mu \in \Gamma_c(M, Vol(M))$

with μ being non-negative

$\downarrow$

• (M, g) satisfies the strong energy condition if $\begin{cases} Ric(X, X) \geq 0 \\ \forall X \in \mathfrak{X}(M) \text{ timelike} \end{cases}$

$(\Longleftrightarrow Ric_{ij} X^i X^j \in \mathcal{D}'(x_\alpha(V_\alpha))$ non negative$)$

(VIII) Regularization of distributions

For $\mathcal{U}_\alpha \subseteq \mathbb{R}^n$, $u \in D'(\mathcal{U}_\alpha)$
and ρ_ε a standard mollifier:

$$u * \rho_\varepsilon(x) := \langle u, \rho_\varepsilon(x - \cdot) \rangle$$

$$\downarrow$$

$$T *_M \rho_\varepsilon := \sum_{\alpha \in \mathbb{N}} \chi_\alpha (\psi_\alpha)^* \left[((\psi_\alpha)_* (\xi_\alpha T)) * \rho_\varepsilon \right]$$

! $u \geq 0 \implies u *_M \rho_\varepsilon \geq 0$!

This construction is crucial to prove the existence of smooth approximations of $g \in \mathcal{C}^0$.

In fact one can prove:

Lemma (8.1)

$\forall\, g \in \mathcal{C}^1$ there exists a net $(\check{g}_\varepsilon)$ of smooth metrics which converge in $\mathcal{C}^1_{loc}$ to g and fulfill:

$$\check{g}_\varepsilon(X,X) \leq 0 \Rightarrow g(X,X) < 0$$

(approximation of lightcones from inside)

By further showing that:

$$\mathrm{Ric}[\check{g}_\varepsilon] - \mathrm{Ric} *_u g_\varepsilon \to 0 \text{ in } \mathcal{C}^0_{loc}$$

the following fundamental Theorem can be proven:

Theorem (8.2) (Melanie Graf)

- (M,g) $\mathcal{C}^1$ - spacetime
- $K \subseteq TM$ compact
- $\mathrm{Ric}(X,X) \geq 0$ $\forall$ timelike $X \in \mathfrak{X}(M)$

$\Longrightarrow \forall \delta > 0 \; \exists \varepsilon_0 > 0 : \forall \varepsilon < \varepsilon_0 \; \forall X \in K$
mit $\check{g}_\varepsilon (X,X) = -1$:

$$\boxed{\mathrm{Ric}\,[\check{g}_\varepsilon]\,(X,X) > -\delta}$$

Theorem 8.3 / C^1-Hawking

(Melanie Graf)

Let (M,g) be a time-orientable Lorentzian manifold with $g \in \mathcal{C}^1$.

If (M,g) :

(1) fulfills the distributional strong energy condition

(non-negative Ricci-curvature)

(2) contains a spacelike Cauchy hypersurface S with

$K(n) > \beta > 0$ on S

then : $\boxed{\begin{array}{l} \tau(S,p) \leq 1/\beta \\ \forall p \in I^+(S) \end{array}}$

$\Rightarrow$ (M,g) is singular

(IX) C^1-Theorem with QEI-inspired energy condition?

Problem : $\mathrm{Ric} \in \mathcal{D}\mathcal{T}^0_2(M)$

$\Rightarrow$ all energy conditions have to be formulated as world volume instead of world line inequalities.

- $\mathcal{Y}[f]$ is dependent on estimates of $(\int_I Ric(X,X) f^2 dt)^*$

↳ a worldvolume inequality does not tell us much about $*$

One special case would be a condition like:

$$Ric(X,X)[f^2] \geq Q \|f^2\|_{L^1(M)}$$

(?)

$$\int Ric[\check{g}_\varepsilon] f^2 \, dVol \geq Q \|f^2\|_{L^1(M)}$$

$\forall f$

$\Rightarrow \quad Ric[\check{g}_\varepsilon] \geq Q$

$\Rightarrow$ Singularity <u>if</u> K big enough ...

<u>What if $g \in C^{1,1}$</u>

$\Rightarrow$ · $\exp^\perp : NS \to M$ bi-Lipschitz homeomorphism
· uniqueness of geodesics

- Radmacher's Theorem $\Rightarrow$ Ric a.e. defined

$\rightarrow$ world line inequalities?

also Stars, shock waves all $C^{1,1}$...