Quiver Varieties

Jakob Schuhmacher

July 6, 2020

Heidelberg University

Quick Recap: Quiver Representations

- A *quiver* $Q = (V, E, s, t)$ is a finite and directed graph.
- A *representation* of Q is a tuple (M, f) consisting of a family $M = (M_k)_{k \in V}$ of complex vector spaces M_k and a family $f = (f_\alpha)_{\alpha \in E}$ of $\mathbb{C}$-linear maps

$$f_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)}.$$

- A representation (M, f) of Q is called *finite dimensional* if there holds

$$\dim_{\mathbb{C}} (M_k) < \infty$$

for all $k \in V$.
- For a finite dimensional representation (M, f) of Q, the vector $\mathbf{m} = \left(\dim_{\mathbb{C}} (M_k)\right)_{k \in V}$ is called the *dimension vector* of (M, f).

The Representation Space

Let $Q = (V, E, s, t)$ be a quiver, and let $M = (M_k)_{k \in V}$ be a family of complex vector spaces M_k.

Definition

The complex vector space

$$R(Q, M) = \bigoplus_{\alpha \in E} \operatorname{Hom}\left(M_{s(\alpha)}, M_{t(\alpha)}\right)$$

is called the *representation space* of Q with respect to M.

Any element $z = (z_\alpha)_{\alpha \in E} \in R(Q, M)$ induces a representation (M, z) of Q.

The Group Action on the Representation Space

Assume now that M_k is finite dimensional for all $k \in V$. We consider the group

$$\mathrm{GL}\left(M\right) = \prod_{k \in V} \mathrm{GL}\left(M_k\right).$$

Note that $\mathrm{GL}\left(M\right)$ is both a complex Lie group and a complex algebraic group.

Theorem

$\mathrm{GL}\left(M\right)$ *acts on* $R\left(Q, M\right)$ *by conjugation. More precisely, this action is given by*

$$g \cdot z = \left(g_{t(\alpha)} z_\alpha g_{s(\alpha)}^{-1}\right)_{\alpha \in E} \in R\left(Q, M\right)$$

for all $g = \left(g_k\right)_{k \in V} \in \mathrm{GL}\left(M\right)$ *and all* $z = \left(z_\alpha\right)_{\alpha \in E} \in R\left(Q, M\right)$*.*

The Double Quiver

Let $|Q| = \left(|V|, |E|\right)$ be a finite graph. Then, we define the *double quiver* $Q = (V, E, s, t)$ of $|Q|$ as follows.

- $V = |V|$,
- $E = H \sqcup \bar{H}$, where $H = \bar{H} = |E|$.
- Define the maps s and t on H in a way such that (V, H, s, t) forms a quiver.
- For any edge $\alpha \in H$, let $\bar{\alpha}$ denote the unique edge in $\bar{H}$ for which there holds $|\alpha| = |\bar{\alpha}| \in |E|$. Then, we define s and t on $\bar{H}$ by $s(\bar{\alpha}) = t(\alpha)$ and $t(\bar{\alpha}) = s(\alpha)$.

More on the Double Quiver

- Let $|Q|$ denote the Kronecker graph.

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \end{tikzcd}$$

Then, the double quiver Q of $|Q|$ is given as follows.

$$\begin{tikzcd} 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \end{tikzcd}$$

- Note that for a double quiver Q, there is a natural direct sum decomposition

$$R(Q, M) = R(H, M) \oplus R\left(\bar{H}, M\right),$$

where

$$R(H, M) = \bigoplus_{\alpha \in H} \operatorname{Hom}\left(M_{s(\alpha)}, M_{t(\alpha)}\right),$$

$$R\left(\bar{H}, M\right) = \bigoplus_{\bar{\alpha} \in \bar{H}} \operatorname{Hom}\left(M_{s(\bar{\alpha})}, M_{t(\bar{\alpha})}\right).$$

Some Notation

- For the remainder of the talk, fix a double quiver $Q = (V, E, s, t)$ and two families $M = (M_k)_{k \in V}$ and $N = (N_k)_{k \in V}$ of finite dimensional complex vector spaces.
- Moreover, we write

$$\operatorname{Hom}(M, N) = \bigoplus_{k \in V} \operatorname{Hom}(M_k, N_k)$$

and

$$\operatorname{Hom}(N, M) = \bigoplus_{k \in V} \operatorname{Hom}(N_k, M_k).$$

Further Group Actions

- $\mathrm{GL}(M)$ acts on $\mathrm{Hom}(M, N)$ by

$$g \cdot \varphi = \varphi \circ g^{-1} \in \mathrm{Hom}(M, N)$$

  for all $g \in \mathrm{GL}(M)$ and $\varphi \in \mathrm{Hom}(M, N)$.

- $\mathrm{GL}(M)$ acts on $\mathrm{Hom}(N, M)$ by

$$g \cdot \psi = g \circ \psi \in \mathrm{Hom}(N, M)$$

  for all $g \in \mathrm{GL}(M)$ and $\psi \in \mathrm{Hom}(N, M)$.

Framed Representation Space

Definition

The complex vector space

$$R(Q, M, N) = R(Q, M) \oplus \mathrm{Hom}(M, N) \oplus \mathrm{Hom}(N, M)$$

is called the N-*framed representation space* of Q with respect to M.

- Since we have defined $\mathrm{GL}(M)$-actions on each of the direct summands of $R(Q, M, N)$, we naturally obtain a $\mathbb{C}$-linear action

$$\mathrm{GL}(M) \curvearrowright R(Q, M, N).$$

- With respect to the direct sum decomposition of $R(Q, M)$, we denote elements by $(x, y, \varphi, \psi) \in R(Q, M, N)$.

An Example of a Framed Representation

- Let Q be the double quiver of the Kronecker graph.

$$\begin{tikzcd} 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \end{tikzcd}$$

- An N-framed representation (x, y, φ, ψ) with respect to M looks as follows.

$$\begin{tikzcd} M_1 \arrow[r, shift left, "x"] \arrow[d, shift left, "\varphi_1"] & M_2 \arrow[l, shift left, "y"] \arrow[d, shift left, "\varphi_2"] \\ N_1 \arrow[u, shift left, "\psi_1"] & N_2 \arrow[u, shift left, "\psi_2"] \end{tikzcd}$$

- In particular, framed representations are representations of another double quiver. In this case, the double quiver of the graph A_4.

Affine GIT Quotients

- Let X be a non-singular affine variety, and let G be a reductive linear algebraic group which acts algebraically on X.
- The variety

$$X /\!\!/ G = \operatorname{maxSpec}\left(\mathbb{C}[X]^G\right)$$

is called the *affine GIT quotient* of X along G.

Theorem (Mumford, 1965)

Let G act freely on X. Then, $X /\!\!/ G$ and X/G are homeomorphic, and $X /\!\!/ G$ is a non-singular variety of dimension $\dim\left(X /\!\!/ G\right) = \dim(X) - \dim(G)$.

Twisted GIT Quotients

- Let X be a non-singular affine variety, let G be a reductive linear algebraic group which acts algebraically on X, and let $\chi\colon G \to \mathbb{G}_m$ be a character.
- We consider the *semi-invariant*

$$\mathbb{C}[X]^{G,\chi} = \left\{ f \in \mathbb{C}[X] \mid f(g \cdot x) = \chi(g) f(x) \text{ for all } g, x \right\}.$$

Definition

The variety

$$X /\!\!/_{\chi} G = \operatorname{Proj}\left(\bigoplus_{n \in \mathbb{N}_0} \mathbb{C}[X]^{G,\chi^n} \right)$$

is called the *twisted GIT quotient* of X along G with respect to χ.

Definition

A tuple (M, g, I, J, K) consisting of a smooth manifold M, a Riemannian metric g on M, and three complex structures I, J, and K is called a *hyperkähler manifold* if the following assertions hold true:

1. (M, g, I), (M, g, J) and (M, g, K) are Kähler manifolds.
2. $I^2 = J^2 = K^2 = IJK = -\operatorname{id}_{TM}$.

- Hyperkähler manifolds are naturally complex symplectic.
- The dimension of hyperkähler manifolds is divisible by 4.
- Example: $\left(\mathbb{H}^n, g_{\mathbb{R}^{4n}}, I, J, K\right)$, where I, J, K are induced by scalar multiplication with $i, j, k \in \mathbb{H}$.

Hyperkähler Quotients

- Let a compact real Lie group C act smoothly on a hyperkähler manifold M.
- Assume $C \curvearrowright M$ preserves the Riemannian metric g and all complex structures I, J, K.
- Assume that the action $C \curvearrowright (M, \omega_S)$ is Hamiltonian with moment map $\mu_S \colon M \to \mathfrak{c}^*$ for all $S \in \{I, J, K\}$.
- This induces the *hyperkähler moment map* $\mu = (\mu_I, \mu_J, \mu_K)$.
- Pick an Ad^*-invariant element $\zeta \in \mathbb{R}^3 \otimes \mathfrak{c}^*$ such that the restricted action $C \curvearrowright \mu^{-1}(\zeta)$ is free.

Theorem (Hitchin et al., 1987)

$\mu^{-1}(\zeta)/C$ naturally is a hyperkähler manifold of real dimension $\dim(M) - 4\dim(C)$.

- Pick hermitian metrics on M_k and N_k for all $k \in V$.
- This induces a hermitian metric h on $R(Q, M, N)$.
- $g = \mathrm{Re}(h)$ is a Riemannian metric on $R(Q, M, N)$.
- Define I as scalar multiplication with $i \in \mathbb{C}$.
- $J(x, y, \varphi, \psi) = (-y^*, x^*, \psi^*, -\varphi^*)$.
- $K = IJ$.

Theorem

$\left(R(Q, M, N), g, I, J, K\right)$ *is a hyperkähler vector space.*

Hyperkähler Quiver Variety (Part 2)

- Due to the hermitian metrics in M_k, we can define the group $\mathrm{U}(M) \subseteq \mathrm{GL}(M)$.
- $\mathrm{U}(M)$ is a real compact Lie group.
- We obtain an action $\mathrm{U}(M) \curvearrowright R(Q, M, N)$, which is Hamiltonian with respect to ω_S for all $S \in \{I, J, K\}$.
- Pick a certain $\zeta \in \mathbb{R}^3 \otimes \mathfrak{u}(M)^*$ (suitable choice needed in order for the restricted action to be free).

Definition

The hyperkähler manifold

$$\mathcal{M}_\zeta(Q, M, N) = \mu^{-1}(\zeta) / \mathrm{U}(M)$$

is called *hyperkähler quiver variety*.

GIT Quiver Variety (Part 1)

Suppose you are given the following data:

- $\mathbb{A}(R)$ denotes the affine complex variety induced by $R(Q, M, N)$.
- A skew-symmetric function $\varepsilon \colon E \to \mathbb{C}^{\times}$, that means that there holds $\varepsilon(\alpha) = -\varepsilon(\bar{\alpha})$ for all $\alpha \in E$.
- $\lambda \in \mathbb{C}^{\mathrm{card}(V)}$.
- $\theta \in \mathbb{Z}^{\mathrm{card}(V)}$.

Then:

- ε defines a symplectic structure ω_ε on $\mathbb{A}(R)$ for which the $\mathrm{GL}(M)$-action is Hamiltonian (moment map: μ_ε).
- θ induces a character $\chi_\theta \colon \mathrm{GL}(M) \to \mathbb{G}_m$.

GIT Quiver Variety (Part 2)

Definition

The quasiprojective variety

$$\mathcal{M}_{\theta}^{\varepsilon,\lambda}(Q, M, N) = \mu_{\varepsilon}^{-1}(\lambda) /\!\!/_{\chi_\theta} \mathrm{GL}(M)$$

is called the *GIT quiver variety.*

- $\mathcal{M}_{\theta}^{\varepsilon,\lambda}(Q, M, N)$ is non-singular. Thus, there exists an associated complex manifold, called the *analytification* of $\mathcal{M}_{\theta}^{\varepsilon,\lambda}(Q, M, N)$, which we denote by

$$\left(\mathcal{M}_{\theta}^{\varepsilon,\lambda}(Q, M, N)\right)^{\mathrm{an}}.$$

- $\mathcal{M}_{0}^{\varepsilon,\lambda}(Q, M, N) = \mu_{\varepsilon}^{-1}(\lambda) /\!\!/ \mathrm{GL}(M)$.

Hyperkähler Quiver Variety vs. GIT Quiver Variety

- Choose the function $\varepsilon\colon E \to \mathbb{C}^{\times}$ as

$$\varepsilon(\alpha) = \begin{cases} 1, \text{ if } \alpha \in H, \\ -1 \text{ if } \alpha \in \bar{H}. \end{cases} .$$

- Consider the hyperkähler GIT quiver variety as a complex manifold $\mathcal{M}_\zeta(Q, M, N)$ with respect to its first complex structure I.

Theorem (Kempf, Ness, Nakajima)

For suitable choices of ζ, θ and λ, there is an isomorphism

$$\mathcal{M}_\zeta(Q, M, N) \cong \left(\mathcal{M}_\theta^{\varepsilon,\lambda}(Q, M, N)\right)^{an}$$

of complex manifolds.

An Example (Part 1)

- Consider the graph $|Q| = A_r$

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & r \end{tikzcd}$$

  and the associated double quiver $Q = (V, E, s, t)$

$$\begin{tikzcd} 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & r \arrow[l, shift left] \end{tikzcd}$$

- Let $N = \bigoplus_{k \in V} N_k$ be given by

$$\begin{aligned} N_1 &= N_2 = \ldots = N_{r-1} = 0, \\ N_r &= \mathbb{C}^n \,. \end{aligned}$$

- Let $M = \bigoplus M_k$ be some $\big(\mathbb{Z} \,/ r\, \mathbb{Z}\big)$-graded vector space with dimension vector

$$\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}_0^r \,.$$

An Example (Part 2)

- An N-framed representation of Q with respect to M is of the following form.

$$M_1 \underset{y_1}{\overset{x_1}{\rightleftarrows}} M_2 \underset{y_2}{\overset{x_2}{\rightleftarrows}} \cdots \underset{y_{r-1}}{\overset{x_{r-1}}{\rightleftarrows}} M_r \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbb{C}^n$$

- Stability theory of quiver representations (King, 1994) yields that there needs to hold

$$0 \leq m_1 \leq m_2 \leq \ldots \leq m_r \leq n$$

in order for the quiver variety $\mathcal{M}_\zeta(Q, M, N)$ to be non-empty.

An Example (Part 3)

- Without loss of generality, we can assume that

$$0 < m_1 < m_2 < \ldots < m_r < n.$$

  Otherwise, we could remove one of the vertices of the quiver without changing the resulting quiver variety.

- Let $\mathcal{F} = \mathcal{F}(\mathbb{C}^n, m_1, \ldots, m_r, n)$ be the flag manifold of all flags in $\mathbb{C}^n$ with prescribed signature $(m_1, \ldots, m_r, n)$, that means elements of $\mathcal{F}$ are sequences of complex vector spaces

$$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots \subseteq E_r \subseteq E_{r+1} = \mathbb{C}^n .$$

- $\mathcal{F}$ is a complex manifold. Thus, $T^*\mathcal{F}$ is a complex symplectic manifold.

An Example (Part 4)

Theorem (Nakajima, 1994)

For a suitable choice of ζ, the hyperkähler quiver variety $\mathcal{M}_\zeta(Q, M, N)$ (considered as a complex manifold with respect to I) is isomorphic to the complex manifold $T^*\mathcal{F}$.