Group actions on hyperbolic spaces

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Introduction

Groups and geometry

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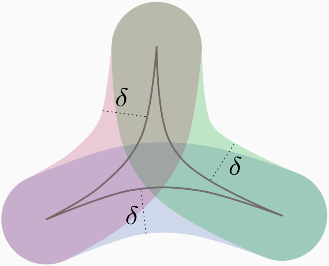
Example(Theorem): If a group G acts freely on a tree, then G is a free group.

Hyperbolic geometry

Hyperbolic spaces

Let $\delta \geq 0$. A geodesic metric space X is δ-hyperbolic if all the triangles are δ-*thin*:



Example/Non-example

Example: Every tree is a 0-hyperbolic space.

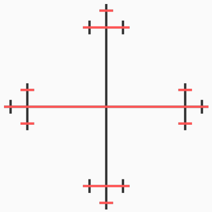


Figure 1: The tree of valence two, also known as the Cayley graph of F_2 or the universal cover of $\mathbb{S}^1 \wedge \mathbb{S}^1$.

Non-example: $\mathbb{R}^2$.(Can you see why?)

Hyperbolic Groups

Hyperbolic groups

A f.g. group G is hyperbolic if one of the two equivalent conditions hold:

- If its *Cayley graph* is hyperbolic. (One vertex for each element of the group, and one edge between g and g'if there exists s in the generating set such that $gg^{-1} = s$). [1]

Properly discontinuous: if for every compact $K \subset X$ (with the metric topology), the set $\{g \in G \mid gK \cap K\}$ is finite.

Cocompact: If the quotient X/G is compact (with the quotient topology).

We call such an action geometric.

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Hyperbolic groups

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- If its *Cayley graph* is hyperbolic. (One vertex for each element of the group, and one edge between g and g'if there exists s in the generating set such that $gg^{-1} = s$). [1]
- If it admits a properly discontinuous, cocompact action by isometries on a hyperbolic space.

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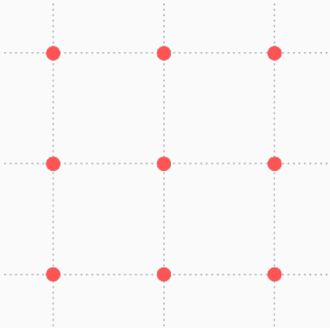
Example/Non-example

Example: F_2 the free group on two generators.

Non-example: $\mathbb{Z}^2$.



Question

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Goal:
Study quotients of (hyperbolic) groups

Small Cancellation Theory

Classic Small Cancellation

Consider S finite and $F(S)$. Let $R \subset F(S)$.

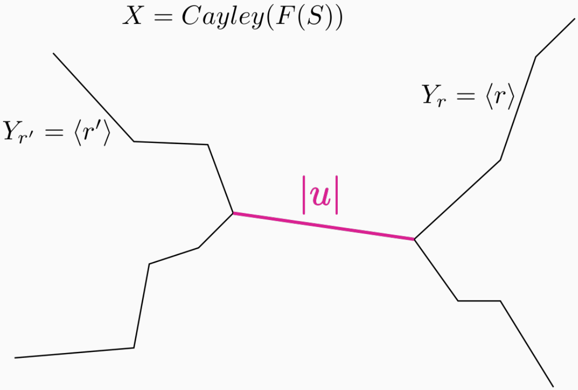
$$F(S) \rightsquigarrow F(S)/\langle\langle R \rangle\rangle =: \overline{G}$$

Let $\lambda > 0$. We say that R satisfies the $C'(\lambda)$-*small cancellation condition* if for all $r \in R$, and for every u piece (a common prefix between two distinct elements of the set of all cyclic conjugates of $R \cup R^{-1}$)

$$|u| \leq \lambda |r|$$

. **Theorem:** Let $\lambda \in (0, \frac{1}{6})$. If R satisfies the $C'(\lambda)$-small cancellation condition, then $\overline{G}$ is hyperbolic.

Classic Small Cancellation



$$|r| = \inf_{x \in X} \{d_X(rx, x)\} =: [r],$$

$$|u| = diam(Y_r \cap Y_{r'}),$$

$$C'(\lambda): \quad \sup_{r \neq r'} \{diam(Y_r \cap Y_{r'})\} \leq \lambda \inf_r \{[r]\}$$

Start with G acting geometrically on a δ hyperbolic space X.

Consider a pair (H, Y):

- $Y \subset X$ 2δ-*quasi-convex*, geodesics starting and ending on it stay close to it.
- $H \trianglelefteq Stab_G(Y)$ acting cocompactly on Y.

Denote $\mathcal{Q} := \{(gHg^{-1}, gY) \mid g \in G\} = \{(H_i, Y_i) \mid i \in \mathcal{I}\}$

$$G \rightsquigarrow \overline{G} = G/K$$

where $K := \langle\langle H_i \mid i \in \mathcal{I}\rangle\rangle$.

WANT: Construct a hypebrolic space $\overline{X}$ on which $\overline{G}$ acts geometrically.

Constructing $\overline{X}$: The Cones

$$X \xrightarrow{\text{Attaching cones } Z(Y_i)} \dot{X}_\rho(\mathcal{Q})$$

Let $\rho > 0$, for every $i \in \mathcal{I}$, the cone *over Y_i of radius ρ* is defined as follows

$$Z(Y_i) := Y_i \times [0, \rho]/(y, 0) \sim (y', 0)$$

it is endowed with a metric $d_{Z(Y_i)}$ such that for $x = (y, t)$ and $x' = (y', t')$ points in $Z(Y_i)$ the following holds:

1. If $t, t' > 0$ and $d_{Y_i}(y, y') < \pi \sinh(\rho)$, then there is a bijection between the set of geodesic segments joining y to y' in Y_i and the set of geodesic segments joining x to x' in $Z(Y_i)$.
2. In all other cases we have $d_{Z(Y_i)}(x, x') = r + r'$. Moreover, there is a unique geodesic connecting x and x' passing through the apex $v := (y, 0)$.

Constructing $\overline{X}$: The Cone-Off

The cone-off over X relative to the family $\mathcal{Q}$

$$\dot{X}_\rho(\mathcal{Q}) := X \sqcup \left(\coprod_{i \in I} Z(Y_i) \right) / \sim$$

where $\sim$ is the equivalence relation that identifies for each $i \in I$ the subspace Y_i and its image on $Z(Y_i)$ under the map $\iota : Y_i \to Z(Y_i)$, such that $y \mapsto (y, \rho)$.

Defining a metric $d_{\dot{X}}$:

1. A metric on the disjoint union.
2. A pseudo-metric on the quotient induced by the previous one.

Warning:

- The attaching maps are not isometries.

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Defining a metric $d_{\dot{X}}$:

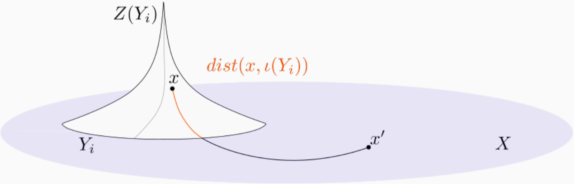
1. A metric on the disjoint union.
2. A pseudo-metric on the quotient induced by the previous one.

Warning:

- The attaching maps are not isometries.
- We want it to be positive definite, i.e. a metric.

Proposition
$(\dot{X}, d_{\dot{X}})$ *is a metric space.*

What happens when a point is in a cone?



If the points are in X nothing is altered.

Constructing $\overline{X}$: Hyperbolicity of the Cone-Off

$$\Delta(\mathcal{Q}) := \sup\left\{\operatorname{diam}(Y_1^{+5\delta} \cap Y_2^{+5\delta}) \mid (H_1, Y_1) \neq (H_2, Y_2) \in \mathcal{Q}\right\},$$
$$T(\mathcal{Q}) := \inf\left\{[h] \mid h \in H \setminus \{1\}, H \in \mathcal{Q}(H)\right\},$$

where where $[h] = \min\{d_X(hy, y) \mid y \in Y\}$ is the translation length of h.

Theorem
There exist universal positive numbers $\rho > 10^{20}\delta_{\mathbb{H}^2}, \delta_0$, and Δ_0 which do not depend on X, G, nor $\mathcal{Q}$, with the following property: If

$$\delta \leq \delta_0 \quad \textit{and} \quad \Delta(\mathcal{Q}) \leq \Delta_0,$$

then the spaces $\dot{X}$ is $\dot{\delta}$-hyperbolic.

Remark
$\dot{\delta}$ only depends on $\delta_{\mathbb{H}^2}$, moreover $\dot{\delta} >> \delta_{\mathbb{H}^2}$.

Constructing $\overline{X}$: The action

$G \curvearrowright X$ extends to $G \curvearrowright \dot{X}$:

$(H, Y) \in \mathcal{Q}$. For every $x = (y, r) \in Z(Y)$, and every $g \in G$, we have

$$(g, x) \mapsto gx = (gy, r) \in Z(gY).$$

For the points in $\dot{X} \setminus \sqcup Z(Y)$ the action remains unaltered. By definition it is also by isometries.

Some feature about the action:

Lemma
Let x be a point in $\dot{X}$ that is in the α-neighbourhood of X. Then the set of elements $g \in G$ such that $d_{\dot{X}}(gx, x) < 2(\rho - \alpha)$ is finite.

Finally constructing $\overline{X}$

We define

$$\overline{X}_\rho(\mathcal{Q}) := \dot{X}/K,$$

and construct the pseudo-metric $d_{\overline{X}}$ which is induced by the metric on the cone-off.

Proposition

If $T(\mathcal{Q}) \geq \pi \sinh(\rho)$*, then* $(\overline{X}, d_{\overline{X}})$ *is a metric space.*

Theorem

There exist universal positive numbers $\rho > 10^{20}\delta_{\mathbb{H}^2}, \delta_0$*, and* Δ_0 *which do not depend on* X*,* G*, nor* $\mathcal{Q}$*, with the following property: If*

$$\delta \leq \delta_0 \quad \textit{and} \quad \Delta(\mathcal{Q}) \leq \Delta_0,$$

then the space $\overline{X}$ *is* $\overline{\delta}$*-hyperbolic.*

Remark

$\overline{\delta}$ *only depends on* $\delta_{\mathbb{H}^2}$*, moreover* $\dot{\delta} >> \delta_{\mathbb{H}^2}$

The action of $\overline{G}$

By definition of the metric, $\overline{G} \curvearrowright \overline{X}$ by isometries as well.

Proposition
The action is geometric.

It suffices to see that $\{\overline{g} \in \overline{G} \mid gB(\overline{x}, \overline{\delta}) \cap B(\overline{x}, \overline{\delta})\}$ is finite.

- Points close to the image of the apices $d_{\overline{X}}(\overline{v}, \overline{x}) < \rho - \overline{\delta}$
- Points that are $\overline{\delta}$-close to the image of $\dot{X}$ on $\overline{X}$.

For cocompactnes we compare $\overline{X}/\overline{G}$ with X/G, and the cones $Z(Y/H)$ with $Z(Y)/H$.

The theorem

Theorem (Small Cancellation Theorem)
There exist positive constants ρ_0, δ_0, and Δ_0 with the following property: Let G be a group acting geometrically on a δ-hyperbolic space X, let be $\mathcal{Q}$ a family of pairs $\{(H_i, Y_i) \mid i \in \mathcal{I}\}$, where Y is quasi-convex, and H a subgroup of G stabilizing Y, and acting cocompactly on it. If

$$\rho \geq \rho_0, \quad \delta \leq \delta_0,$$
$$\Delta(\mathcal{Q}) \leq \Delta_0, \quad \textit{and} \quad T(\mathcal{Q}) \geq \pi \sinh(\rho).$$

Then, the group $\overline{G}$ is hyperbolic.

Thank you!

More on hyperbolicity

Theorem (Cartan-Hadamard theorem)
Let $\delta' \geq 0$ and $\sigma > 10^7\delta'$. Let X' be a length spaces. If every ball of radius σ of X' is δ' hyperbolic and X is $10^{-5}\sigma$-simply-connected, i.e its fundamental group is normally generated by free homotopies of loops whose diameter less than $10^{-5}\sigma$, then X' is $300\delta'$-hyperbolic.

Hyperbolicity of $\dot{X}$

Let $\rho > 10^{20}\delta_{\mathbb{H}^2}$.

Lemma
Every ball of radius $\sigma > 10^8\delta_{\mathbb{H}^2}$ is $3\delta_{\mathbb{H}^2}$-hyperbolic

Lemma
$\dot{X}$ is 40δ-simply connected

If $\delta \leq \delta_{\mathbb{H}^2}$, then we can apply Cartan-Hadamard. It is by assumptions on the theorem (δ_0).

Proposition
$\dot{X}$ is $900\delta_{\mathbb{H}^2}$ hyperbolic.

Hyperbolicity of $\overline{X}$

Here it is crucial that $T(\mathcal{Q}) \geq \pi \sinh(\rho)$.

Let $\sigma = 2\rho$, so it also depends only on $\delta_{\mathbb{H}^2}$.

Lemma
Every ball of radius $\frac{\rho}{20}$ on $\overline{X}$ is $2\dot{\delta}$-hyperbolic.

Lemma
$\overline{X}$ is $40\dot{\delta}$-simply-connected.

Proposition
$\overline{X}$ is $600\dot{\delta}$-hyperbolic, i.e. it is $54 \times 10^4 \delta_{\mathbb{H}^2}$-hyperbolic.