An introduction to opers

A fine class of objects

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Notation

We fix the following notations. Let

- X be a compact Riemann surface of genus $g \geq 2$
- E be a holomorphic vector bundle over X
- $\{F_i\}$ be a filtration of holomorphic subbundles of E
- ∇ be a holomorphic connection on E

Classical opers as vector bundles

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    for K_X the canonical bundle of X.
- An $\mathrm{SL}(n, \mathbb{C})$-oper has an additional isomorphism $\det(E) \simeq \mathcal{O}_X$.

Classical opers as vector bundles

- An $\mathrm{Sp}(2n, \mathbb{C})$-oper is an $\mathrm{SL}(2n, \mathbb{C})$-oper with a horizontal symplectic form on E, compatible with $\det(E) \simeq \mathcal{O}_X$, such that $F_i^\perp = F_{n-i}$.
- An $\mathrm{SO}(2n+1, \mathbb{C})$-oper is an $\mathrm{SL}(2n+1, \mathbb{C})$-oper with a horizontal nondegenerate symmetric bilinear form on E, compatible with $\det(E) \simeq \mathcal{O}_X$, such that $F_i^\perp = F_{n-i}$.

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Remark

A flat holomorphic bundle can admit at most 1 filtration that satisfies the conditions of an $\mathrm{SL}(n, \mathbb{C})$-oper.

Notation for principal bundle

Let

- G be a complex (semi)simple Lie group (simply-connected or adjoint-type)
- B be a choice of Borel subgroup in G
- ω be a holomorphic connection on some principal G-bundle

Classical G-opers

Definition

A *G-oper* on a X is (E_G, E_B, ω) where

- E_G is a principal G-bundle
- E_B is a principal B-bundle and a holomorphic reduction of structure group of E_G
- ω is a holomorphic connection on E_G compatible with the reduction of structure group.

i.e. ω satisfies Griffiths transversality and nondegeneracy.

A fine set of objects

Theorem (Teleman, Hejhal, Drinfeld–Sokolev...

The set of G-oper structures on a fixed G-bundle forms an affine space modelled on the Hitchin base

$$\mathcal{B} = \bigoplus_{i=1}^{n-1} H^0(X, K^{i+1})$$

(as a vector space)

Parameterization of opers

We can see this using Lie theoretic data from a principal $\mathfrak{sl}_2$ embedding:

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When $G = \mathrm{SL}(n, \mathbb{C})$ and on some fixed G-bundle we may parameterize the opers as

$$\nabla_u = d + \hbar^{-1}(X_- + \sum_{i=1}^{n-1} P_i(z) X_i) dz.$$

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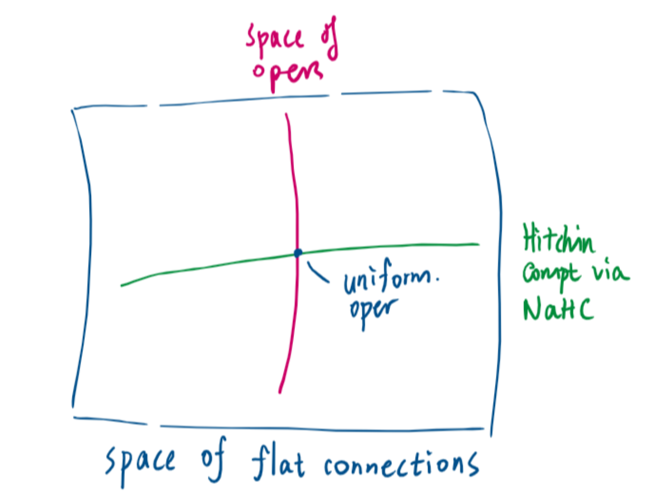
Remark

The non-abelian Hodge correspondence sends this Higgs field to the flat connection:

$$\varphi_u \mapsto D_h + \varphi_u + \varphi_u^{*_h}.$$

Two Lagrangian subspaces



Opers as complex projective structures

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Proposition (Gunning '66)

Complex projective structures (up to marked isomorphism) form an affine space on $H^0(X, K^2)$. As a corollary they are $\mathrm{PSL}(2, \mathbb{C})$-opers.

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To show this we can apply the Schwarzian derivative to the projective coordinates and get a holomorphic quadratic differential. The construction globalizes due to the Möbius invariance of the Schwarzian.

Opers as differential operators

The Hitchin base can also be related to certain linear differential equations of order n on $\widetilde{X}$ with regular singularities satisfying some invariance under change of coordinates. Hence opers can also be viewed as differential operators (see DFK+ '21, Hejhal '75).

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For example for a fixed $K^{1/2}$, there is a correspondence between

$$\nabla = \partial_z + \hbar^{-1} \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} dz \quad \leftrightarrow \quad D = q(z) - \hbar^2 \partial_z^2$$

where ∇ is a flat connection on

$$0 \to K^{1/2} \to E \to K^{-1/2} \to 0$$

and D is a Schrödinger operator on sections of $K^{-1/2}$.

Opers as jet bundles

Another way to identify opers as differential operators is via jet bundles, which record the Taylor expansion of sections in a coordinate-free way.

Theorem (Biswas '03)

There is a natural isomorphism between E and the $(n-1)$th jet bundle on the last associated graded bundle

$$E \simeq J^{n-1}(F_n/F_{n-1}),$$

and the isomorphism sends $\{F_i\}$ to a natural filtration of the jet bundle.

$\hbar$-conformal limit

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It provides a new identification between the Hitchin component and the space of opers as transverse Lagrangians in the space of flat connections.

Generalizing opers

Definition (Collier-Sanders '21)

A (G, P)-oper on X is (E_G, E_P, ω) where

- E_G is a holomorphic principal G-bundle on X
- E_P is a holomorphic reduction to the parabolic subgroup $P < G$
- ω is a holomorphic connection on E_G compatible with the reduction of structure group.

As before the compatibility criteria are based on Griffiths transversality and nondegeneracy.

Parametrization of higher Teichmüller spaces

There is a parameter space shared by (G,P)-opers and higher Teichmüller spaces. (Collier–Sanders '21, BCG+ '21)

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Inside the moduli space of Higgs bundles, we get a generalization of the Hitchin components called Cayley components. By non-abelian Hodge correspondence these are higher Teichmüller spaces.

Future directions

- Identify a conformal limit type of correspondence for (G, P)-opers arising from a magical $\mathfrak{sl}_2$-triple?
- Generalize the Hitchin map for the Cayley components to its parameter space?
- Generalize the complex projective structure description of $\mathrm{PSL}(2, \mathbb{C})$-opers to higher rank?