Seminar Lie algebras and representations
Problem sheet 5: The adjoint representation and the classification of semisimple Lie algebras.

We saw in today's lecture that to each semisimple Lie algebra $\mathfrak{g}$ we can associate a Dynkin diagram as follows:

Step 1: choose a sytem of simple roots $S = \{\alpha_1, \ldots, \alpha_n\}$, and let E be its real span. Clearly, the roots $R \subset E$.

Step 2: by identifying $\mathfrak{g} \cong \mathfrak{g}^*$, we see that the Killing form B induces a scalar product on E. Denote it by $(\,\cdot\,,\,\cdot\,)$.

Step 3: the scalar product satisfies that for any roots $\alpha, \beta \in R$, we have $n_{\beta\alpha} := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, so that $n_{\beta\alpha}n_{\alpha\beta} = 4\cos^2\theta_{\alpha\beta}$, with $\theta_{\alpha\beta}$ is the angle between the vectors $\alpha, \beta \in E$.

Step 4: draw the diagram with vertices $\mathcal{V} = \{v_{\alpha_i} \;:\; i = 1, \ldots, n\}$ corresponding simple roots and $-n_{\beta\alpha}$ edges between them if $\pi/2 < \theta_{\alpha\beta} < \pi$ and $||\beta|| \geq ||\alpha||$. Moreover, if $||\beta|| \gneq ||\alpha||$, we draw a $>$ sign between both edges (opening towards the vertex corresponding to the longest root).

For $n = 1$, we recover the root system of $\mathfrak{sl}(2, \mathbb{C})$; for $n = 2$, we have the root systems of $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{so}(5, \mathbb{C})$ and the following diagram:

Assume there exists a Lie algebra whose Dynkin diagram is the one given above (this is in fact the case, and it is called $\mathfrak{g}_2$). Explain how to recover its bracket multiplication table following the steps indicated below:

1. Choose generators for the Lie subalgebras $\mathfrak{sl}(\alpha_i)$, $i = 1, 2$, say H_i, X_i, Y_i such that
$$[X_i, Y_i] = H_i, \quad [H_i, X_i] = 2X_i \quad [H_i, Y_i] = -2Y_i.$$

2. Check there are 6 positive roots. You will need to use condition (4) in page 320 of Fulton–Harris, an their decomposition as sums of simple roots. This gives the dimension of the Lie algebra (14).

3. Use the above to choose a generator X_α for each eigenspace $(\mathfrak{g}_2)_\alpha$.

4. Decompose $\mathfrak{g}_2$ according to the action of $\mathfrak{sl}(\alpha_i)$.

5. Give an example illustrating how to compute all other brackets.