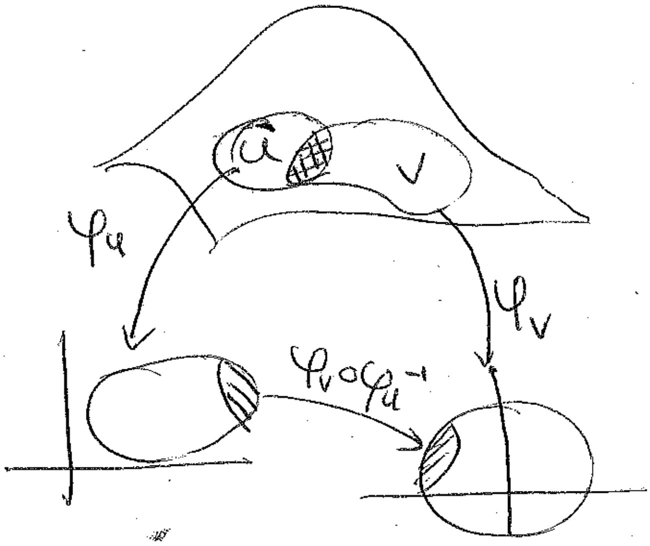
SCHEMES

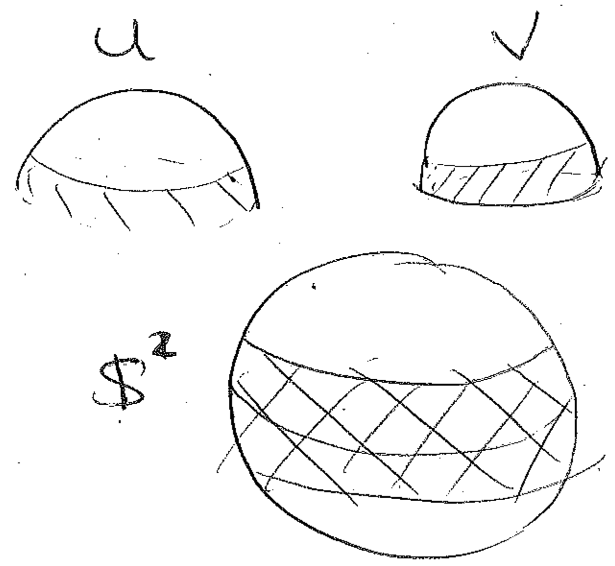
1 Why schemes?

Motivation 1: manifolds are defined to be locally isomorphic to $\mathbb{C}^n / \mathbb{R}^n$ & the different pieces are pasted together by means of differentiable / holom. maps

$\mathbb{P}^n$



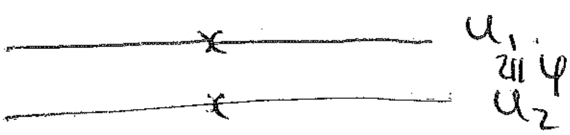
S^2



S^2

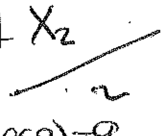
Now, algebraic varieties do not paste to algebraic varieties (affine or proj.)

Indeed: take $X_1 = \mathbb{A}^1 = X_2$, let $U_i = X_i \setminus \{0\} \subset X_i$



Glue X_1 & X_2 along $U_1 \cong U_2$

$X = X_1 \amalg X_2 \big/ \sim$

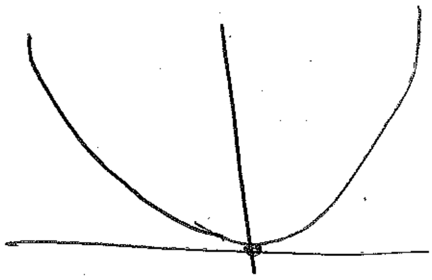


$p \sim q \Leftrightarrow \varphi(p) = q$

X is not a variety, not even locally, as it is non-separated

Motivation 2

Consider the Line $\{y=0\}=L$ and the conic $C=\{y=x^2\}$. Their set theoretic intersection is $(0,0)$.



Now, the minimal ideal containing both $\langle y\rangle$ and $\langle y-x^2\rangle$ is $\langle x^2, y\rangle$. It vanishes at $\bar{0}$, but it contains much more information: a tangent vector in the direction $y=0$.

$$\frac{\mathbb{K}[x,y]}{\langle x^2,y\rangle} = \mathbb{K}\oplus x\mathbb{K}$$

Note that whereas $\langle y-\lambda x\rangle \subset \langle x,y\rangle$ $\forall \lambda\in\mathbb{K}$ (i.e., $\bar{0}\subset L_\lambda=\{y-\lambda x=0\}$), the curve L is the only one containing $(0,0)$ and having $\frac{\partial}{\partial x}$ as a tangent vector (i.e. $\langle y-\lambda x\rangle \subset \langle x^2,y\rangle \Leftrightarrow \lambda=0$.

This points out that the appropriate intersection is $\mathrm{Spec}\left(\frac{\mathbb{K}[x,y]}{\langle x^2,y\rangle}\right)$.

Motivation 3

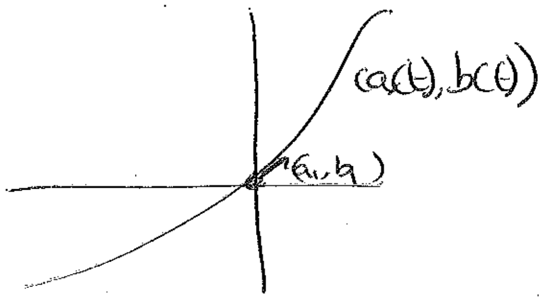
Let $X=\{(0,0),(a,b)\} = \mathrm{Spec}\left(\frac{\mathbb{K}[x,y]}{\langle x,y\rangle\cap\langle x-a, x-b\rangle}\right)$

$$=\mathbb{K}[x,y]/\langle x^2-ax, xy-bx, xy-ay, y^2-by\rangle$$

Now, consider the family

$X_t=\{(0,0),(a(t),b(t))\}$ $\quad a(t)=tp(t) = t(a_1+a_2t^2+\dots)$ $\quad b(t)=tq(t) = t(b_1+b_2t^2+\dots)$

$I_t = \langle x,y\rangle \cap \langle a(t), b(t)\rangle$



By continuity of multiplication

$I_t \longrightarrow I_0$ is an ideal

$$I_t = \langle x^2 - a(t)x,\ xy - b(t)x,\ xy - a(t)y,\ y^2 - b(t)y \rangle$$

$$\downarrow$$

$$I_0 \supseteq \langle x^2, xy, y^2 \rangle$$

Also $\dfrac{a(t)y - b(t)x}{t} \in I_t$

$$\downarrow$$

$$a_1 y - b_1 x \in I_0$$

<u>Now</u>:

$$\left.\begin{array}{c} \operatorname{Codim}_{k[x,y]} I_t = 2 \\ \parallel \\ \operatorname{Codim}_{k[x,y]} \langle x^2, xy, y^2, a_1 y - b_1 x \rangle \end{array}\right\} \Longrightarrow I_0 = \langle x^2, xy, y^2, a_1 y - b_1 x \rangle$$

<u>Motivation 4</u> (Local schemes) The Zariski topology is way too coarse: In $\mathbb{K}[x,y]$ the closest we can get to a point preserving some geometry is via $D_{m_x} = \bigcap_{\substack{x \in U \\ U \text{ Zariski}}} U$

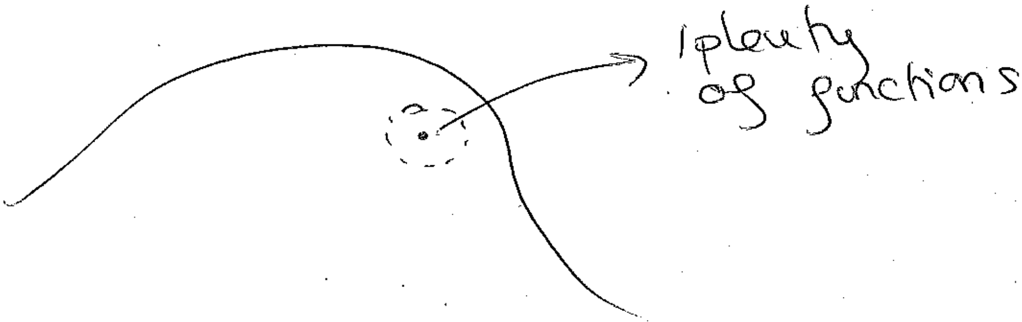
Nevertheless $\operatorname{Spec}(\mathbb{K}[x,y]_{(x,y)})$, has too much "noise", as it contains not only the closed point $\langle x,y\rangle$, but also irreducible varieties containing $(0,0)$. This implies that for $\neq$ closed points $\operatorname{Spec}(\mathbb{K}[x,y]_{m_x}) \not\cong \operatorname{Spec}(\mathbb{K}[x,y]_{m_{x'}})$ (that is, these local neighbourhoods contain too much global information).

We can instead consider $\mathbb{K}[[x,y]] = \{\sum_I a_I x^{i_1} y^{i_2}\}$ the ring of formal series

$$\mathbb{K}[x,y] \hookrightarrow \mathbb{K}[x,y]_{\langle x,y\rangle} \hookrightarrow \mathbb{K}[[x,y]]$$

$$\mathbb{A}^2_{\mathbb{K}} \longleftarrow \operatorname{Spec}(\mathbb{K}[x,y]_{m_x}) \longleftarrow \operatorname{Spec}(\mathbb{K}[[x,y]])$$

Think of $\mathbb{K}[[x,y]]$ as the ring of ~~Taylor~~ expansions around $(0,0)$. These should be functions defined on a sufficiently small neighbourhood. Recall that we wanted to make a parallelism between geometric spaces and functions on them. In a sense, functions contain all the info we need. The closests into the space we look, the more functions we get.



② Formal definition

__Def__: an affine scheme is $(X, \mathcal{O}_X)$ where $X = \operatorname{Spec} R$, $\mathcal{O}_X$ is a sheaf of rings whose stalks are local rings.

* We will think of affine schemes as varieties with their structure sheaf and some open sets in it (D_f).

<u>Def</u> : a scheme is a locally ringed space $(X, \mathcal{O}_X)$ s.t. $\forall x \in X \quad \exists\, U \ni x$ s.t. $U = \mathrm{Spec}(R)$, $\mathcal{O}_X|_U = \mathcal{O}_U$

<u>Examples</u>

1) $\left(\mathbb{A}^2_{\mathbb{K}} \setminus \{0\},\ \overbrace{\mathcal{O}_{\mathbb{A}^2}}^{1}\Big|_{\mathbb{A}^2 \setminus \{0\}}\right)$

$$\mathbb{A}^2_{\mathbb{K}} \setminus \{0\} = \{x_1 \neq 0\} \cup \{x_2 \neq 0\} = D_{x_1} \cup D_{x_2}$$

Regular functions at a point of an open neighb. are germs, so their definition depends only on open subsets around points. Namely $\mathcal{O}|_{D_{x_i}} = \mathcal{O}_{D_{x_i}}$.

2) Glueing: generalising the example

———— : ————, we may consider any two varieties X, Y with open sets

$U \subset X \quad V \subset Y$ which are mapped isomorphically to one another

$U \xrightarrow{f} V$

is such a way that

$$\mathcal{O}_Y|_V \xrightarrow{f^\#} \mathcal{O}_X|_U$$

is an isomorphism.

Then, we define a scheme by taking

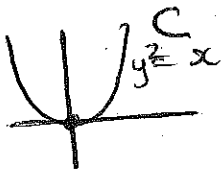
$Z = X \amalg Y / \sim \qquad x \sim y \Leftrightarrow \begin{cases} x \in X, y \in Y \\ \text{and } f(x) = y \end{cases}$

As for $\mathcal{O}_{Z}(V) = \{ \langle s_x, s_y \} \quad s_x \in \mathcal{O}(f_1^{-1}(V)), s_y \in \mathcal{O}(i_2^{-1}(V))$

$$s_x|_U \circ f = s_y|_V$$

3) Projective space is a scheme

4) $\mathrm{Spec}\left(\mathbb{R}[x,y]/\langle x^3 y \rangle\right)$ is the scheme theoretic intersection $C \cap L$



$L = \{y=0\}$

③ Some more on sheaves

Before we can classify schemes, we need to have a closer look at sheaves.

Let X be a topological space, and let $\mathcal{F}$ be a presheaf of rings on it. That is

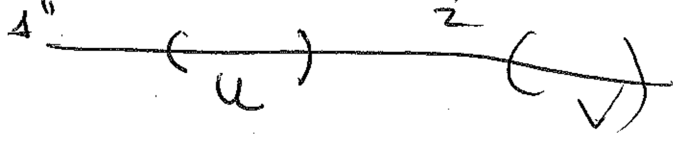
$$\mathcal{F} : \mathrm{top}\, X \longrightarrow \mathrm{Rings}$$
$$U \longmapsto \mathcal{F}(U)$$
$$V \overset{i}{\hookrightarrow} U \longmapsto \rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$$

withe the natural coherence conditions given by (PS 1) – (PS 4)

Example

1) $\overline{\mathbb{C}}(U) = \mathbb{C}$. Think of them as constant functions on $U \subset \mathbb{C}$.

Note that $\exists\ s_1 \in \overline{\mathbb{C}}(U) \quad s_2 \in \overline{\mathbb{C}}(V)$



that do not glue to any constant function on $U \cup V$.

2) Let $\mathcal{F}(U)$ = continuous functions on $U \subseteq \mathbb{C}$ open

Let $\rho^U_V = \begin{cases} id & \text{if } U = V \\ 0 & \text{otherwise} \end{cases}$

Then, any function is allowed on an open set U, but its restriction to a covering not containing U will be zero.

<u>So</u> there are presheaves which are not sheaves.

However, any presheaf $\mathcal{F}$ has a sheafification, that is a sheaf $\tilde{\mathcal{F}}$ which is the sheaf that's closest to the presheaf $\mathcal{F}$.

- In example 1) $\tilde{\mathbb{C}} =: \underline{\mathbb{C}}$ consists of locally constant functions.
- In example 2) $\tilde{\mathcal{F}} = 0$.

Formal definition:

* $\mathcal{F}$ presheaf $\rightsquigarrow$ $\mathcal{F}_x$ stalks $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$
* $\tilde{\mathcal{F}}(U) = \{ s : U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x \ , \ s(x) \in \mathcal{F}_x \}$

Examples

1) $\mathbb{C}_x = \mathbb{C}$ $\quad \tilde{\mathbb{C}}(U) = \{ s : U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x \}$ locally ct-functions

2) $\mathcal{F}$ as in 2) before: $\mathcal{F}_x = 0 \quad \forall x$.

We would like for the sheafification of a sheaf to be the same sheaf. For this we define:

<u>Def</u>: Let $\mathcal{F}$, $\mathcal{G}$ be ~~sheaves~~ of rings on X topological space. A morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of maps

$$\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for all $U \subset X$ open, such that φ_U is a ring homom. and:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho^U_V}\downarrow & \circlearrowright & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V) \end{array}$$

A morphism is an isomorphism s.t. $\exists$ an inverse

<u>Theorem</u>: $\mathcal{F}$ sheaf $\Rightarrow$ $\mathcal{F} \cong \tilde{\mathcal{F}}$

Given a morphism of sheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$, one can consider its kernel, image and cokernel

- $\mathrm{Ker}(\varphi)(U) = \mathrm{Ker}(\varphi_U)$ is a sheaf
- $\mathrm{Im}(\varphi)(U) = \mathrm{Im}(\varphi_U)$ is only a presheaf
- $\mathrm{Coker}(\varphi)(U) = \mathrm{Coker}(\varphi_U) = \dfrac{\mathcal{G}(U)}{\mathrm{Im}(\varphi_U)}$ "

because $\mathrm{Im}\,\varphi$ is not a sheaf.

Example:

On $\mathbb{R}^2 \setminus 0$ consider $\mathcal{C}^\infty(U) = \left\{ \begin{array}{c} \text{differentiable} \\ \text{functions} \\ \text{on } U \end{array} \right\}$

$d\mathcal{C}^\infty(U) = \{ \text{exact forms on } U : df \quad f \in C^\infty(U) \}$
$= \mathrm{Im}(d : \mathcal{C}^\infty \longrightarrow \Omega^1)$

Now, on any simply connected closed subset all forms are exact, but

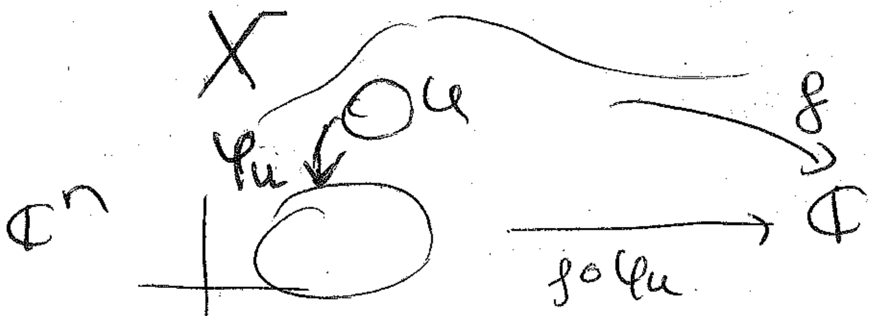
$$\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x\, dy}{x^2+y^2}$$

is not, inspite of $\exists$ covering (by open balls, for example) on which $\omega_{|U} = d f_U$. In other words, axiom (S2) is violated.

Def $\widetilde{\mathrm{Im}}(\varphi)$ is the sheafification of the presheaf $\mathrm{Im}(\varphi)(U) = \mathrm{Im}(\varphi_U)$. We will drop the tilde from the notation. Likewise for coker.

Examples 1) On M connected manifold $/\mathbb{C}$. $\mathcal{O}_X$ holomorphic functions on X, $\mathcal{O}_X^*$ nowhere vanishing holomorphic functions.

Holomorphic functions $f : X \longrightarrow \mathbb{C}$ are such that around each point z $\exists\, U \ni z$ open s.t. $f \circ \varphi_U = F(z)$ can be expressed as a converging series.



$F(z) = \sum_I a_I z^I$

Consider $\mathcal{O}_X \xrightarrow{\exp(2\pi i\,\cdot\,)} \mathcal{O}_X^\times$

$$\mathcal{O}_X(\mathcal{U}) \longrightarrow \mathcal{O}_X^\times(\mathcal{U})$$
$$f \longmapsto \exp(2\pi i f)$$

We can think of $\mathcal{O}_X(\mathcal{U})$ ($\mathcal{O}_X^\times(\mathcal{U})$) as equivalence classes of sets of series $\{s_{\mathcal{U}_i} = \sum_I z_0^I a_I\}_{\{\mathcal{U}_i \to \mathcal{U}\}\ \text{covering}}$ where $s_{\mathcal{U}_i}|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_{\mathcal{U}_j}|_{\mathcal{U}_i \cap \mathcal{U}_j}$ (and $s_{\mathcal{U}_i}(z) \neq 0$ for $\mathcal{O}_X^\times$) modulo

$$\{s_{\mathcal{U}_i}\}_{\{\mathcal{U}_i\}} \sim \{s_{\mathcal{U}'_i}\}_{\{\mathcal{U}'_i\}} \iff s_{\mathcal{U}_i}|_{\mathcal{U}_i \cap \mathcal{U}'_j} = s_{\mathcal{U}'_j}|_{\mathcal{U}_i \cap \mathcal{U}'_j}$$

(All this means is that holomorphicity is local)

So it yields that $\exp(2\pi i\,\cdot)$ is surjective.
Nevertheless, take $\mathcal{U} \subset X$ a non simply connected open set. Then

$$\mathcal{O}_X(\mathcal{U}) \not\twoheadrightarrow \mathcal{O}_X^\times(\mathcal{U})$$

e.g. on $\mathbb{C}^2 \setminus 0$ we need 2 open sets $\mathcal{U} = \mathbb{C}^2 \setminus \{0\vec{x}\}$, $V = \mathbb{C}^2 \setminus \{0\vec{y}\}$ to recover all non vanishing holo functions on $\mathbb{C}^2 \setminus 0$ (logarithm.)

What is the kernel of $\exp(2\pi i(\cdot))$?

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X^\times$$

(This is a short exact sequence of sheaves)

WARNING: in this case $\mathcal{O}_X / \mathcal{I} : U \mapsto \mathcal{O}_X(U) / \mathcal{O}_{\mathcal{I}}(U)$ is a sheaf. But quotients of sheaves need not be sheaves. For example:

2) $X = \mathbb{P}^1_{\mathbb{C}}$ $\qquad \mathcal{F} = \mathcal{O}_X$, $\mathcal{G} = \mathcal{O}_X(-0)$

$U = X \setminus \{0\}$

$V = X \setminus \{\infty\}$

$$\mathcal{G} \hookrightarrow \mathcal{F}$$
$$f(z) \longmapsto \frac{f(z)}{z} \quad \text{(unique extension}$$

Consider on $U \cap V$ the map

$$\mathcal{F}|_{U\cap V} \longrightarrow \mathcal{G}|_{U\cap V}$$
$$f(z) \longmapsto z f(z)$$

Note that $\mathcal{F}|_{U\cap V} \simeq \mathcal{G}|_{U\cap V}$.

So let $\frac{\mathcal{F}}{\mathcal{G}}(W) = \mathcal{F}(W) / \mathcal{G}(W)$. Let

$$f(z) = \begin{cases} [z] & \text{if } z \in U \\ [c] & \text{if } z \in V \end{cases} \qquad c \in \mathbb{C}^\times$$

Then $[z]|_{U\cap V} = 0$ as $\mathcal{F}|_{U\cap V} \cong \mathcal{G}|_{U\cap V}$

$[c]|_{U\cap V} = 0$

but $z - c \notin \mathcal{G}(U\cap V)$ as $z-c$ does not vanish at $0 \Rightarrow [z]$ and $[c]$ do not glue to $\mathcal{F}(X) / \mathcal{G}(X)$

Prop: $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves. Then φ is an isomorphism $\Leftrightarrow$ $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is an isomorphism of stalks.

Recall that

$$\mathcal{F}_p = \varinjlim_{p \in \mathcal{U}} \mathcal{F}(\mathcal{U}) = \left\{ \underset{\mathcal{F}(\mathcal{U})}{(s_{\mathcal{U}})}_{p\in\mathcal{U}} \quad \begin{array}{c} s_{\mathcal{U}}|_W = s_V|_W \\ \text{for some } W \subset \mathcal{U}\cap V \end{array} \right\}$$

Hence, if we call $\mathcal{F}(\mathcal{U})$ the sections of the sheaf, $\mathcal{F}_p$ are germs of sections of $\mathcal{F}$ at p.

Rk: $\mathcal{F}(\mathcal{U})$ are in fact sections of some projection $\mathbf{F} \xrightarrow{\pi} X$. $\mathbf{F}$ is the space étalé of $\mathcal{F}$. As a set it is just $\mathbf{F} = \bigcup \mathcal{F}_p$

$\pi : \mathbf{F} \longrightarrow X$. For each section $s \in \mathcal{F}(\mathcal{U})$

$s_p \in \mathcal{F}_p \mapsto p$

we build one of π by $\bar{s}(p) = s_p$ the germ of s at p. Clearly $\pi \circ \bar{s} = id$. $\mathbf{F}$ can be endowed with a topology making both π and $\bar{s}$ continuous.

*In particular, $\begin{cases} \varphi \text{ is surjective} \Leftrightarrow \operatorname{Im}\varphi = \mathcal{G} \text{ (the sheaf!)} \\ \varphi \text{ is injective} \Leftrightarrow \operatorname{Ker}\varphi = 0 \end{cases}$

as $\operatorname{Im}\varphi = \mathcal{G} \Leftrightarrow \varphi_p(\mathcal{F}_p) = \mathcal{G}_p$

$\operatorname{Ker}\varphi = 0 \Leftrightarrow \operatorname{Ker}\varphi_p = 0$

4 Back to schemes

With the above material we can now make our statements about schemes precise. Due to lack of time let me leave it as an exercise (!) (I will add it to some proper notes to appear) and mention instead a huge theorem letting us move to the analytic setting without guilt.

Theorem (Chow, Serre): every analytic subspace of $\mathbb{P}^n$ is algebraic (namely, it can be locally described as the vanishing locus of some homogeneous polynomials inside $\mathbb{P}^n$) In other words, any scheme yields an analytic space and viceversa.

Moreover, $\{$coherent sheaves on $X\}$

$$\cong$$

$\{$coherent sheaves on $X_{an}\}$

where X is a scheme, X_{an} its associated analytic space and coherence is defined w.r.t. $\mathcal{O}_X$ (regular functions) and $\mathcal{O}^{an}_{X^{an}}$ (holomorphic functions.)

Recall: a sheaf $\mathcal{F}$ of $\mathcal{O}_X$ ($\mathcal{O}^{an}_{X^{an}}$) modules is coherent if $\exists\, U_i \to X$ covering (by affine sets for X) s.t.

- $\mathcal{F}|_{U_i}$ is a sheaf whose stalks are $M_{i\,m_x}$ where M_i is an $\mathcal{O}_{U_i}$ module $m_x = \{f \in \mathcal{O}_X : f(x)=0\}$
- $\mathcal{F}|_{U_i}$ is finitely generated

Examples 1) Sheaves of sections of v.b.

2) Pushforwards of the same.

A locally free sheaf $\mathcal{F}$ on X (X^{an}) is a sheaf that is locally isomorphic to $\mathcal{O}_X^{\oplus n}$ ($\mathcal{O}_{X^{an}}^{\oplus n}$).