PROJECTIVE VARIETIES

MOTIVATION: Quantum mechanics

* States of physical system $\rightsquigarrow$ Hilbert space $\mathcal{H}$
* Measurements $\rightsquigarrow$ observable operators $\mathcal{O}$
* Measuring $\Rightarrow$ system collapses to eigenstate of $\mathcal{O}$
* Eigenstates provide a basis for $\mathcal{H}$
* Two states $\psi, \lambda\psi \rightsquigarrow$ same physical system
* Consider $\mathcal{H}/\sim \qquad \psi \sim \psi' \Leftrightarrow \psi' = \lambda\psi$.

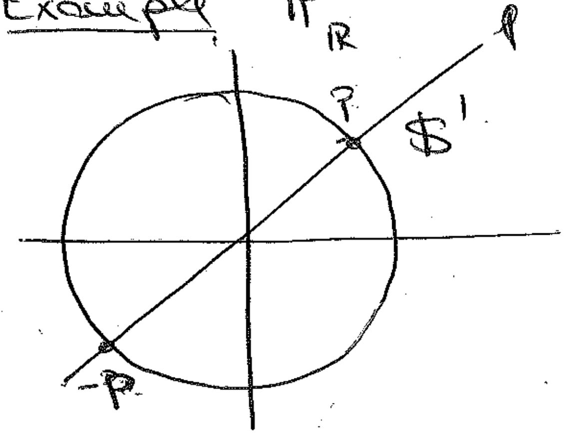
$$\mathcal{H}/\sim \; = \; \mathbb{P}(\mathcal{H})$$

(1) Reminder of differential-geometric construction

Def: $\mathbb{P}^n_{\mathbb{R}} = \{ \ell \subset \mathbb{R}^{n+1} : \ell \text{ vectorial line} \}$

Constructions and coordinates

Example $\mathbb{P}^1_{\mathbb{R}}$



Each line intersects $\mathbb{S}^1$ at 2 points $\{p, -p\}$. Namely, ℓ is totally determined by a point in $\mathbb{S}^1/\mathbb{Z}_2$

Likewise $\mathbb{P}^n_{\mathbb{R}} \cong \mathbb{S}^n/\mathbb{Z}_2$

In other words, a line in $\mathbb{R}^{n+1}$ through the origin is fully determined by a vector in $\mathbb{R}^{n+1}\setminus\{0\}$.
The vector is not unique, only up to multiplication by scalars $\lambda \in \mathbb{R}^{\times}$. Thus

$$\ell \in \mathbb{P}^n_{\mathbb{R}} \iff \underbrace{[a_0 : a_1 : \cdots : a_n]}_{\frac{\mathbb{R}^{n+1}\setminus\{0\}}{\mathbb{R}^*}} = \{\lambda(a_0, \dots, a_n) : \lambda \in \mathbb{R}^*\}$$

Locally $\cong \mathbb{R}^n$: let $U_i \subset \mathbb{P}^n_{\mathbb{R}}$

$$U_i = \{\bar{a} : a_i \neq 0\} = \left\{\left[\frac{a_0}{a_i} : \cdots : \frac{a_{i-1}}{a_i} : 1 : \frac{a_{i+1}}{a_i} : \cdots \frac{a_n}{a_i}\right]\right\} \cong \mathbb{R}^{n+1}$$

$\mathbb{P}^n_{\mathbb{R}} = \bigcup_{i=0} U_i$ covered by affine spaces. This is a property of <u>schemes</u>.

The same construction holds for $\mathbb{P}^n_{\mathbb{C}}$.

2) Algebraic construction

<u>Problem</u> polynomials are not well defined on $\mathbb{P}^n_{\mathbb{K}}$.
To define projective algebraic sets $\rightsquigarrow$?

If $f(a_1, \dots, a_n) = f(\lambda a_1, \dots, \lambda a_n)$ $\forall \bar{a}$ $\forall \lambda$ $\Rightarrow f \equiv c$

<u>However</u> f homogeneous $\rightsquigarrow$ $f = 0$ is well defined on $\mathbb{P}^n_{\mathbb{K}}$

e.g. $f(x_0, x_1) = x_0^2 + x_1 x_0 + 5x_1^2$ $\qquad f(\lambda x_0, \lambda x_1) = \lambda^2 f(x_0, x_1)$

Hence $f(x_0, x_1) = 0 \iff f(\lambda x_0, \lambda x_1) = 0$ $\qquad \lambda \in \mathbb{K}^{\times} := \mathbb{K}\setminus\{0\}$

~~NOT~~ Let $\mathbb{K}[x_0, \dots, x_n]_d$ be the subset of degree d polys.
Given $f \in \mathbb{K}[x_0, \dots, x_n]$ we write $f^{(d)}$ for its degree d part. Namely $f = \sum_{d=0}^{\deg f} f^{(d)}$.

Def: a projective algebraic set $V(I)$ is the zero set of $I \subset \mathbb{K}[x_0, \dots, x_n]$ homogeneous ideal (i.e., an ideal grated by homogeneous polynomials) inside $\mathbb{P}^n_{\mathbb{K}}$.

Remarks: I homogeneous $\iff \forall f \in I \quad f^{(d)} \in I$. Exercise
e.g. $\langle x+y^2, x-y^2 \rangle = \langle x, y^2 \rangle$

Examples

1) $L \subset \mathbb{A}^{n+1}_{\mathbb{K}}$ $k+1$ dimensional linear subspace
$L = \{x_0 = \dots = x_{n-k} = 0\}$

$\bar{L} := L/_{\mathbb{K}^\times} \subset \mathbb{P}^n_{\mathbb{K}}$ is called a linear subspace of $\mathbb{P}^n_{\mathbb{K}}$

In coordinates $\bar{L} = \{[\overset{n-k}{0 : \dots : 0} : a_{n-k+1} : \dots : a_n]\}$

$\xrightarrow[\text{defined}]{\text{to be}} \cong \mathbb{P}^k_{\mathbb{K}}$

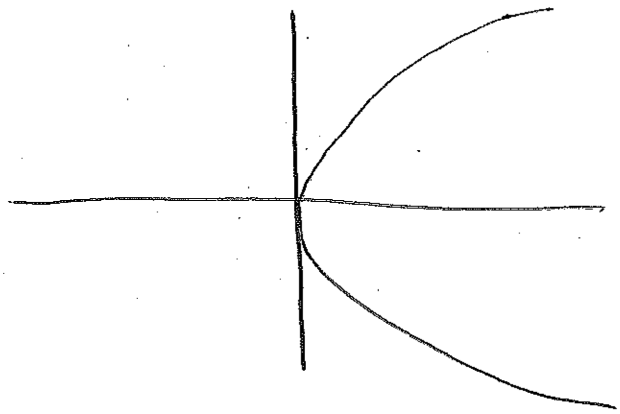
That is, affine varieties induce projective varieties as long as defining ideal homogeneous. To distinguish
$V_{aff}(I) \subset \mathbb{A}^{n+1}_{\mathbb{K}} \qquad V_{proj}(I) \subset \mathbb{P}^n_{\mathbb{K}}$
$I \subset \mathbb{K}[x_0, \dots, x_n]$ homogeneous

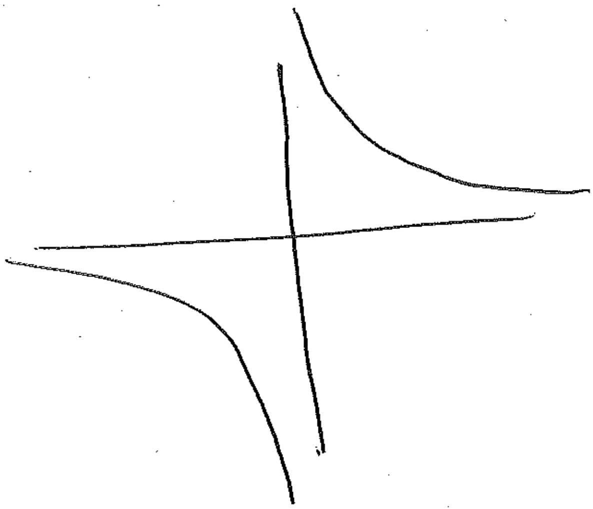
$V_{aff}(I)$ is what we call the cone of $V_{proj}(I)$, and denote it by $C(V_{proj}(I))$.

~~Examp~~

(2) Homogenization

$X_1 = \{x = y^2\}$ $\qquad\qquad$ $X_2 = \{xy = 1\}$





$X_i \subset \mathbb{A}^2_{\mathbb{k}} \subset \mathbb{P}^2_{\mathbb{k}}$

$\tilde{X}_1 = \{zx = y^2\}$ $\qquad\qquad$ $\tilde{X}_2 = \{xy = z^2\}$

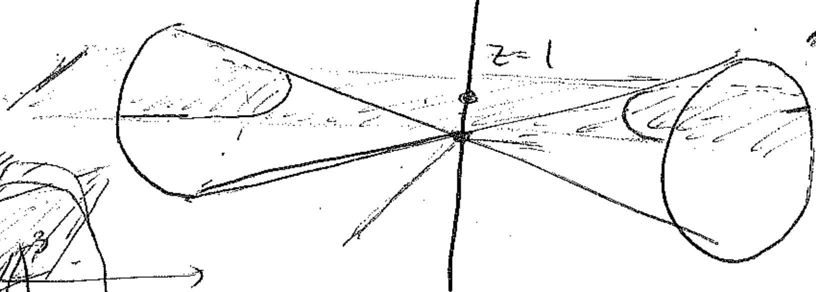
Hence $X_i = \tilde{X}_i \cap \{z \neq 0\}$

Now, $\tilde{X}_1 \cap \{z=0\} = \{[1:0:0]\}$ (direction in which $X_1 \to \infty$)

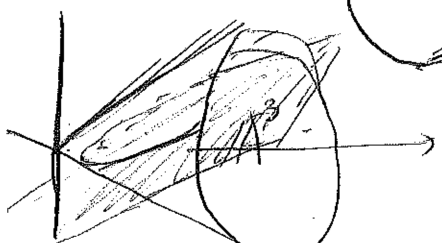
$\tilde{X}_2 \cap \{z=0\} = \{[1:0:0], [0:1:0]\}$ (directions in which $X_2 \to \infty$)

<u>NOTE</u> $\tilde{X}_1, \tilde{X}_2$ are the same up to a permutation of the constants. That is $\tilde{X}_1 \cap \{z \neq 0\} \cong \tilde{X}_2 \cap \{y \neq 0\}$

$\tilde{X}_1 \cap \{z=0\} \cong \tilde{X}_2 \cap \{y=0\}$



$\tilde{X}_2 \cap \mathbb{A}^2_{\mathbb{k}} = \{z \neq 0\}$

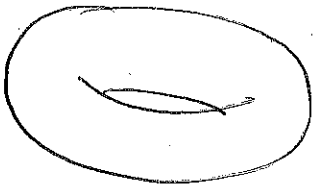


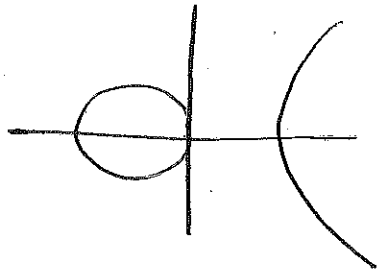
$\tilde{X}_2 \cap \{z \neq 0\}$ $\quad$ Intersect w/ $y = $

(3) $y^2 = x(x^2-1) = x^3 - x$

$\rightsquigarrow$

$y^2 z = x^3 - xz^2$



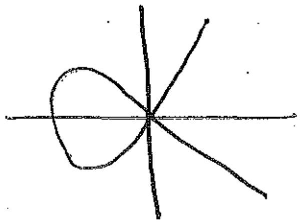


Real locus

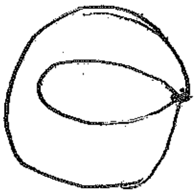
$y^2 = x^2(x+1)$

$\rightsquigarrow$

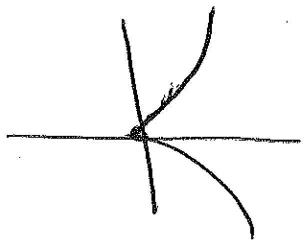
$y^2 z = x^2(x+z)$



Node



$y^2 = x^3$



$\boxed{\text{No 3D Picture available}}$

Cusp

(4) $Y \subset \mathbb{A}^3$ $\qquad Y = \{(t, t^2, t^3)\}$ $\quad t \in \mathbb{K}$

$I(Y) = \langle x^2 - y, \ x^3 - z \rangle$ $\qquad \tilde{Y} \subset \mathbb{P}^3$ has

$I(\tilde{Y}) = \langle x^2 - yw, \ x^3 - zw^2 \rangle = \left\langle w^2\left[\left(\frac{x}{w}\right)^2 - \frac{y}{w}\right], \ w^3 \cdot \left(\left(\frac{x}{w}\right)^3 - \frac{z}{w}\right) \right\rangle$

Using cones, we have a projective Nullstellensatz:
over $\mathbb{C}$

$$\left\{ \begin{array}{c} \text{algebraic sets} \\ \text{in } \mathbb{P}^n_{\mathbb{C}} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{homogeneous radical} \\ \text{ideals in } \mathbb{C}[x_0, \dots, x_n] \\ \text{other than } \langle x_0, \dots, x_n \rangle \end{array} \right\}$$

Gradings

What makes $X \subset \mathbb{P}^n_{\mathbb{C}}$ projective in an intrinsic way?

- Degree induces a grading on $\mathbb{K}[x_0, \dots, x_n] = R$. This ring w/ extra information of grading has more structure.
- If I is homogeneous ideal in $R \rightsquigarrow R/I$ is also graded by degree, as $\deg[p^{(d)}]$ is well defined.

Example

$$\mathbb{K}[x,y] / \langle xy \rangle, \quad p(x,y)^{(d)} = \sum a_{ij} x^i y^j \quad d \geq 2$$

$$p(x,y)^{(d)} \bmod (xy) = a_{d0} x^d + a_{0d} y^d$$

Def: R ring. A grading is a decomposition

$$R = \bigoplus_{j \in J} R_j \qquad R_j \text{ abelian group}$$

$$R_j R_i = R_{i+j} \qquad (J \text{ semigroup})$$

Example $\mathbb{C}[x_0, \dots, x_n]$ admits different gradings. We assign to x_i the degree d_i. In this case

$x_i + x_j^{d_i}$ is homogeneous of degree d_i if $d_j = 1$.

More concretely

$\mathbb{P}(1,1,2)$

degree	1	2	3	
monomials	x, y	x^2, y^2, z	x^3, xy^2, xz, y^3, yz	...

Rk: usual grading in $R = \mathbb{C}[x_0, \dots x_n]$ is defined by taking a $\mathbb{C}^\times$ action on R defined by

$$z * p(x_0, \dots, x_n) = p(z^{-1}x_0, \dots, z^{-1}x_n)$$

Then, $\deg p = k \Leftrightarrow z * p(x_0, \dots, x_n) = z^{-k} p(x_0, \dots, x_n)$

In a similar way $\mathbb{P}(d_0, \dots, d_n)$ has associated grading given by

$$z * p(x_0, \dots x_n) = p(z^{-d_0} x_0, \dots, z^{-d_n} x_n)$$

And $\deg p = k$ in the new grading $\Leftrightarrow z * p = z^{-k} p$

With this: it is easy to see

Exercise: give a construction of $\mathbb{P}_{\mathbb{C}}(1,1,2)$ as a quotient.

Rk: different gradings may give rise to isomorphic projective varieties (e.g. $\mathbb{P}_{\mathbb{C}}(3,3,3) \cong \mathbb{P}^2_{\mathbb{C}}$)

Solution to exercise

$$\mathbb{P}_{\mathbb{C}}(1,1,2) = \frac{\mathbb{C}^3 \setminus \{0\}}{\mathbb{C}^*} \qquad \text{where } \lambda * (x_0, x_1, x_2) = (\lambda x_0, \lambda x_1, \lambda^2 x_2)$$

Note that $\mathbb{P}(1,1,2) \xrightarrow{j} \mathbb{P}^3 \qquad [z_0 : z_1 : z_2 : z_3]$

$$[x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_0 x_1 : x_1^2 : x_2]$$

is a cone with cone point $[0:0:0:1]$.

Coordinates

$$\{z_0 \neq 0\} \cap \operatorname{Im} j = \left\{ \left[1 : \frac{x_1}{x_0} : \left(\frac{x_1}{x_0}\right)^2 : x_2 \right] \right\} \cong \mathbb{A}^2$$

Likewise for $\{z_2 \neq 0\} \cap \operatorname{Im} j$. Note that $\{z_1 \neq 0\} \cap \operatorname{Im} j$ is contained in $\{z_0 \neq 0\} \cap \{z_2 \neq 0\} \cap \operatorname{Im} j$, so it adds no new information.

However, $\{z_3 \neq 0\}$ intersects $\operatorname{Im}(j)$ in a way that no preferred point of an orbit exists. Indeed, one orbit is $[0:0:0:1]$, for which there is no ambiguity. However, for $(\{z_0 \neq 0\} \cup \{z_2 \neq 0\}) \cap \operatorname{Im} j$, all other orbits have two points $\left[\frac{x_0^2}{x_2} : \pm\frac{x_0 x_1}{x_2} : \frac{x_1^2}{x_2} : 1\right]$. No preferred choice can be done. This is an example of an orbifold

The Zariski topology on $\mathbb{P}^n_{\mathbb{K}}$

Consider $\mathbb{K}[\overline{x}]$ with the usual grading, and let $I \subset \mathbb{K}[\overline{x}]$ be homogeneous.

Then, if we let $X = V(I)$, the Zariski topology on X is the one whose closed sets are

* $V(J) \qquad J \supset I$ homogeneous ideal $\quad J_i J_k$

Check: if J_i $i \in I$, homogeneous $\Rightarrow$ so is $\underset{\langle f_1^i \dots f_{n_i}^i \,:\, i \in I\rangle}{\underset{\shortparallel}{\sum J_i}}$ & $J_i \cap J_k$

3) Projective varieties as ringed spaces

$\mathbb{P}^n_{\mathbb{C}}$ is compact in the usual topology. This implies it has no non-constant holomorphic functions on it (Liouville's thm). However, locally there is plenty of them.

The same holds in the algebraic setting. $\mathbb{P}^n_{\mathbb{K}}$ can be endowed with a ring of regular functions

$$\mathcal{O}_{\mathbb{P}^n, p} = \left\{ \frac{f}{g} \quad g(p) \neq 0, \ f, g \in \mathbb{K}[x_0 : \dots : x_n]_d \right\}$$

NOTE $\frac{f}{g}([x_0 : \dots : x_n])$ is well defined as

$$\frac{f}{g}(\lambda \bar{x}) = \frac{\lambda^d}{\lambda^d} \frac{f}{g}(\bar{x}) = \frac{f}{g}(\bar{x})$$

$$\mathcal{O}_{\mathbb{P}}(U) = \bigcap_{p \in X} \mathcal{O}_{X,p} \qquad U \text{ Zariski open} = V_p(I)^c$$

In particular $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}) = \mathbb{K}$ Indeed $g(p) \neq 0 \ \forall p$, $\Rightarrow g \equiv c$ constant $\Rightarrow \deg f = \deg g = 0 \Rightarrow f$ constant.

Prop: $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf (sheaf of regular funct)

The same way, we produce $\mathcal{O}_X$ for $X \subset \mathbb{P}^n$ projective.

Prop: $(X, \mathcal{O}_X)$ is a ringed space locally isomorphic to an affine ringed space.

e.g. $\mathbb{P}^n_{\mathbb{C}} = \bigcup_{i=0}^{n} \mathbb{A}^n_i$ $\quad \mathbb{A}^n_i = \{z_i \neq 0\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\!\wr$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbb{A}^n$

$\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}\big|_{\mathbb{A}^n_i}$ defined by

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}\big|_{\mathbb{A}^n_i}(\mathcal{U}) = \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathcal{U}) \qquad \mathcal{U} \subset \mathbb{A}^n_i$$

$$\| $$

$$\left\{ \frac{f}{g}\Big|_{\mathcal{U}} \text{ homogeneous of the same degree}, \; g|_{\mathcal{U}} \neq 0 \right\}$$

$$\| $$

$$\left\{ \frac{f\left(\frac{z_0}{z_i}, \ldots, 1, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)}{g\left(\frac{z_0}{z_i}, \ldots, 1, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)}, \; f, g \in \mathbb{C}[x_{0,i}, \ldots, x_{n,i}], \; g|_{\mathcal{U}} \neq 0 \right\}$$

$$\| $$

$$\mathcal{O}_{\mathbb{A}^n_i}(\mathcal{U})$$

Rk : this is a particular example of a scheme, in a sense prototypical. Scheme theory unifies affine varieties with locally affine spaces.

Def $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ projective varieties. A morphism $(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$ is a continuous map

$$f : X \longrightarrow Y$$

inducing a $\mathbb{K}$ algebra morphism.

$$f^\# : \mathcal{O}_Y(\mathcal{U}) \longrightarrow \mathcal{O}_X(f^{-1}(\mathcal{U}))$$

An isomorphism is a morphism that admits an inverse.

Examples

1) Regular functions $\mathcal{O}_X(X) \leftrightarrow$ morphisms $X \longrightarrow \mathbb{A}^1$. We will see that these can only be constant.

2) $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^2$

$[s:t] \mapsto [s^2 : st : t^2]$

Continuous, well defined, and induces

$$f^{\#} : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U)) \quad \text{by pullback}$$

$$X := f(\mathbb{P}^1) = V(xz - y^2)$$

$\subseteq)$ obvious

$\supseteq)$ $\forall [x:y:z] \in V(xz-y^2) \Rightarrow x \neq 0$ or $z \neq 0$

If $x \neq 0 \rightsquigarrow (x, y) \mapsto [x^2 : xy : y^2] = [x^2 : xy : xz] = \boxed{[x:y:z]}$

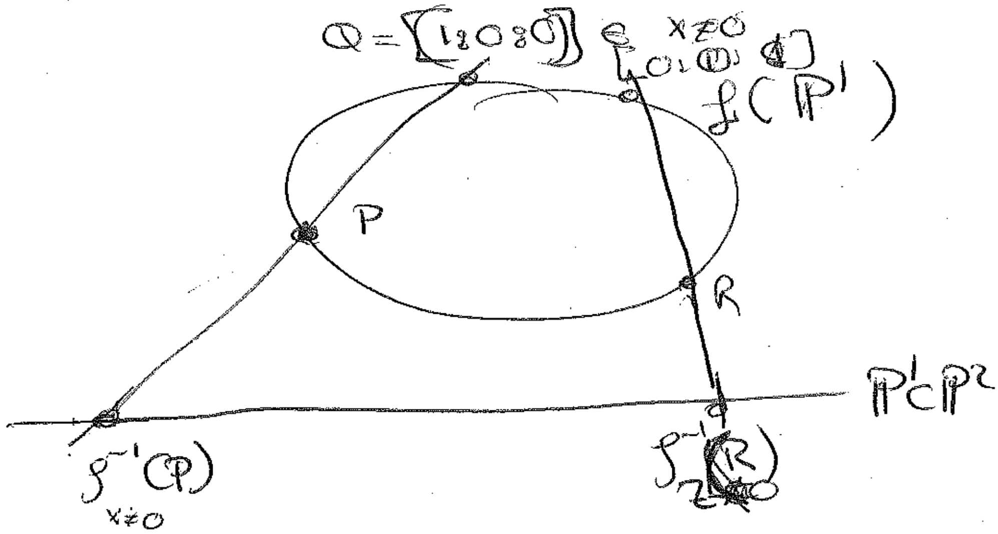
If $z \neq 0 \quad (y, z) \mapsto [y^2 : zy : z^2] = [zx : zy : z^2] = [x:y:z]$

From the above it is easy to see $\exists f^{-1}$ morphism (just need to check that

$$\begin{aligned} f(\mathbb{P}^1) &\longrightarrow \mathbb{P}^1 \\ [x:y:z] &\longmapsto [x:y] \\ [x:y:z] &\longmapsto [y:z] \end{aligned}$$

match on $x \neq 0 \cap z \neq 0$)

Geometrically



3) $\mathbb{P}^N \times \mathbb{P}^M \xrightarrow{f} \mathbb{P}^{(N+1)(M+1)-1}$

$[x_1 \cdots x_N, y_1 \cdots y_M] \mapsto [\underset{z_{11}}{x_1 y_1} : \cdots x_1 y_M : x \cdots : \underset{z_{N,M}}{x_N y_M}]$

Well defined

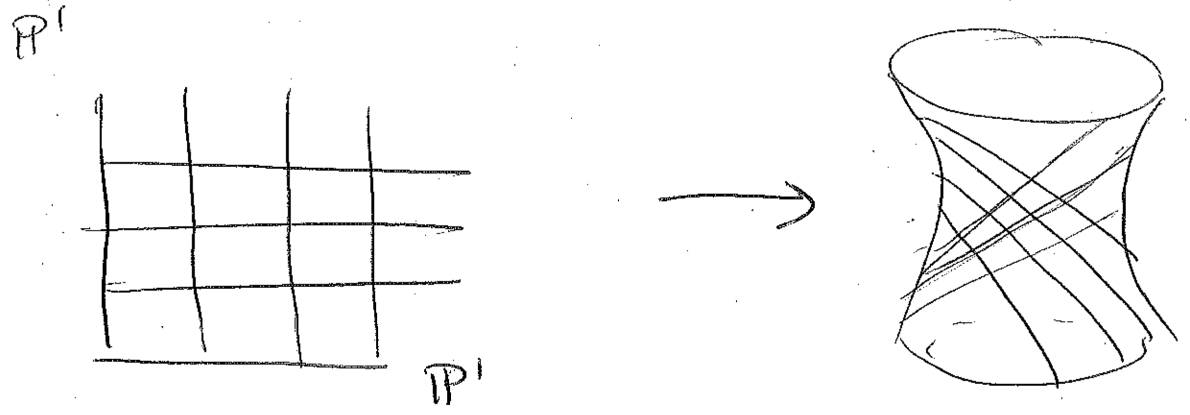
$$\mathrm{Im}(f) = V(z_{i,j} z_{i'j'} - z_{ij'} z_{i'j})$$

Segre embedding.

4) $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$([x_0 : x_1] \times [y_0 : y_1]) \longrightarrow [\underset{z_0}{x_0 y_0} : \underset{z_1}{x_0 y_1} : \underset{z_2}{x_1 y_0} : \underset{z_3}{x_1 y_1}]$

$$V(z_0 z_3 - z_1 z_2) = f(\mathbb{P}^1 \times \mathbb{P}^1)$$



Completeness

Rk in the usual topology $X \subset \mathbb{P}^N_{\mathbb{C}}$ alg. set is compact.

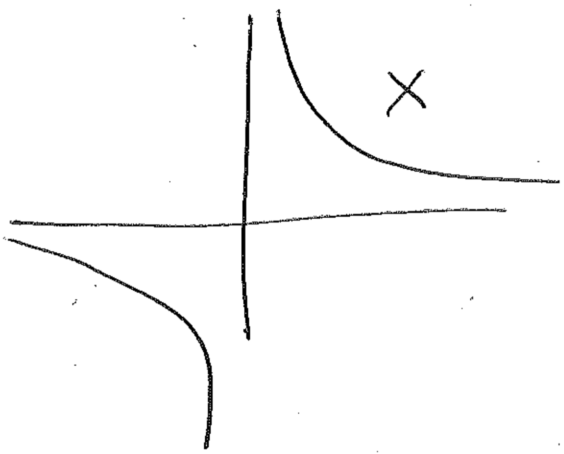
For $\mathbb{P}^N_{\mathbb{C}} = S^{N+1}_{\mathbb{C}} / \mathbb{Z}_2$

For $X \subset \mathbb{P}^N_{\mathbb{C}}$ $\rightsquigarrow$ closed in $\mathbb{P}^N_{\mathbb{C}}$ ✓

→ How do we define compactness? In the Zariski topology even $\mathbb{A}^n$ is compact...

→ A property we may want to emulate is that K compact $K \xrightarrow{f} X$ continuous $\Rightarrow$ $f(K) \subset X$ compact. In our case, closed sets should map to closed sets

Example: The above property does not hold in affine space. Indeed $\underbrace{\{xy = 1\}}_{X=} \subset \mathbb{A}^2$



$$\mathbb{A}^2 \longrightarrow \mathbb{A}^1$$
$$(x,y) \longrightarrow x$$

But Im $X = \mathbb{A}^1 \setminus \{0\}$ open.

Thm $\mathbb{P}^N \times Y \xrightarrow{\pi_2} Y$ is closed (namely, if $X \subset \mathbb{P}^N \times Y$ is closed so is $\pi(X)$) for any variety (affine or projective) Y.

Coro $Z \subset \mathbb{P}^N$ proyective $\Rightarrow$ $Z \times Y \xrightarrow[(*)]{} Y$ is closed.

Def: a variety satisfying (*) is called complete.

Consequences 1) If X is complete, Y any variety then $f: X \to Y$ has closed image

$$\left(\text{take} \quad \begin{array}{ccccc} X & \hookrightarrow & X \times Y & \longrightarrow & Y \\ x & \longmapsto & (x, f(x)) & \longrightarrow & f(x) \end{array} \right)$$

$\Gamma(f) = (x, f(x))$ is closed as it is $f^{-1}(\Delta(Y))^c$ $\underset{X \times Y}{\cap}$

2) If $X \subset \mathbb{P}^n$ var. $X \neq \mathbb{P}^0 \Rightarrow \forall f \in \mathbb{C}[x_0, \dots x_n]_d$ $d > 0$

$X \cap Z(f) \neq \emptyset$

$\Downarrow$

3) $X \xrightarrow{f} \mathbb{A}^1$ is constant $\forall X \subset \mathbb{P}^n$ var.

See f as $f: X \longrightarrow \mathbb{P}^1$ morphism

$f(X) \subsetneq \mathbb{P}^1$

X irreducible $\Rightarrow f(X)$ too

But irreducible subvarieties of $\mathbb{P}^1$ are $\mathbb{P}^1$ and a point.

Examples

2) Does not hold for affine varieties

$\{x = 0\} \subset \mathbb{R}^2$ $\quad \{x - 1 = 0\} \cap \{x = 0\} = \emptyset$

3) Obviously does not hold.

(4) Projective spectrum of a graded ring

<u>Example</u>: what are the irreducible subvarieties of $\mathbb{P}^1_{\mathbb{C}}$? They correspond to prime ideals of $R = \mathbb{C}[x_0, x_1]$

- $\langle x_0, x_1 \rangle$ <u>NOT</u> allowed; $\langle x_0, x_1 \rangle = R_+ = \bigoplus_{d>0} \mathbb{C}[x_0, x_1]_d$
- Assume $\mathfrak{P} = \langle f \rangle$. We claim that $f = a x_0 + b x_1$.

Otherwise

$V_0 = V(\langle f \rangle) \cap \{x_0 \neq 0\}$ is | a union of points $\Rightarrow$ NOT irreducible | empty

If $V_0 = \emptyset \Rightarrow V_1 \neq \emptyset$ is a union of points

- Can you argue why $\mathfrak{P} = \langle f_1^{d_1}, f_2^{d_2} \rangle$ is not possible either? (Hint: assume $1 < d_1 \leq d_2$ and use arguments for $\mathbb{A}^1$)

<u>Def</u>: R graded ring $R = \bigoplus_{d=0}^{\infty} R_d$

$$\mathrm{Proj}(R) = \left\{ \mathfrak{P} \subset R \text{ homogeneous prime ideal} \;\; \mathfrak{P} \not\supseteq R_+ = \bigoplus_{d>0} R_d \right\}$$

<u>Rk</u>: the definition allows rings such as

$\dfrac{\mathbb{C}[x_0, x_1]}{\langle x_0^2 \rangle}$ (Also $\mathrm{Proj}\left(\dfrac{\mathbb{C}[x]}{x^2}\right) = \emptyset$)

Note however that $\mathrm{Proj}\, \dfrac{\mathbb{C}[x_0, x_1]}{\langle x_0^2 \rangle} = \mathrm{Proj}\left(\dfrac{\mathbb{C}[x_0, x_1]}{\langle x_0 \rangle}\right)$

as $x_0^2 \in \mathfrak{P} \Leftrightarrow x_0 \in \mathfrak{P}$.

Thus, there is no 1-1 correspondence between $\mathrm{Proj}(R) \longrightarrow \mathbb{P}^n$ and ideals for $R = \dfrac{\mathbb{C}[\bar{x}]}{I}$

<u>Def</u> : $I \subset \mathbb{C}[x_0 \dots x_n] = R$ homogeneous.

$\overline{I} = \{ p \in R : \exists m \text{ s.t. } x_i^m \cdot p \in I \ \forall i = 0 \dots n \}$
(saturation)

<u>Lemma</u> :

1) $\overline{I}$ is homogeneous
2) $\mathrm{Proj}\ R/I = \mathrm{Proj}\ R/\overline{I}$
3) $\mathrm{Proj}\ R/I = \mathrm{Proj}\ R/J \iff \overline{I} = \overline{J}$
4) $I^{(d)} = \overline{I}^{(d)} \quad d \gg 0$

<u>Consequence</u>

$$\{ \mathrm{Proj}\,(R) \subset \mathrm{Proj}\,(\mathbb{C}[x_0 - x_n]) \} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} I \subset \mathrm{Proj}\ \mathbb{C}[\underline{x}] \\ \text{saturated} \end{matrix} \right\}$$

<u>Rk</u> $\mathrm{Proj}\,(\mathbb{C}[\varepsilon]) = \emptyset$ as $\overline{I} = \langle 1 \rangle$

The Zariski topology is defined as in the affine case, and likewise for the sheaf of functions.

(5) Dimension revisited

We defined the dimension of affine spaces in terms of prime ideals.

The case of projective space should, a priory, not change, as dimension should be (and is) a local property. The main difference is that completeness of projective varieties allow a heavier machinery we do not have in the case of affine varieties. We will compare varieties with one another to compute dimension.

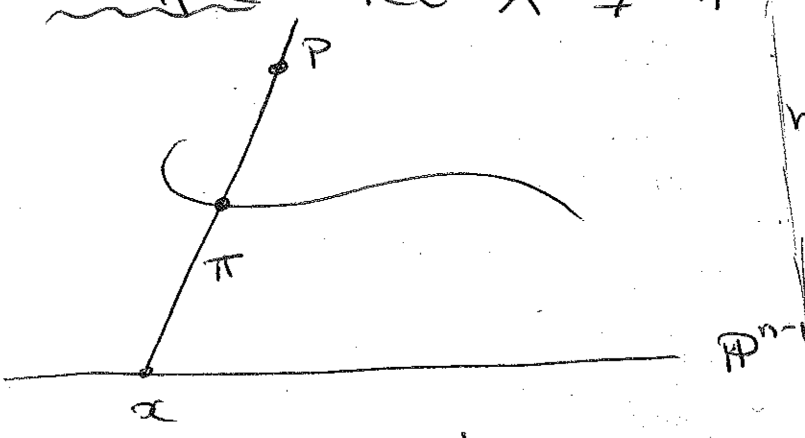
Basic intuition comparison $\leftrightarrow$ morphisms

$f : X \longrightarrow Y$ $\qquad f^{-1}(y) = \{x_1 \cdots x_{n_y}\}$ finite

we should have $\dim X \leq \dim Y$

$f : X \twoheadrightarrow Y$ $\qquad$ surjective $\rightsquigarrow$ $\dim X \geq \dim Y$

Example: let $X \subsetneq \mathbb{P}^n$ projective variety, $P \notin X$.



Choosing an arbitrary hyperplane $\mathbb{P}^{n-1}$: $P \notin \mathbb{P}^{n-1}$ we can project X from P getting a morphism (parametric equations of a line)

Now $\forall\, x \in \mathbb{P}^{n-1}$ $\qquad \pi^{-1}(x) \in \overline{Px} \cong \mathbb{P}^1$ algebraic $\qquad P \notin \pi^{-1}(x) \Rightarrow \pi^{-1}(x)$ finite.

Rk if $P = [0 : \cdots : 0 : 1]$ $\Rightarrow$ $\pi([a_0 : a_1 \cdots : a_n]) = [a_0 : \cdots : a_{n-1}]$

Hence $\dim X = \dim \mathrm{Im}(\pi)$ (intuitively)

If $\exists\, p \in \mathbb{P}^{n-1} \setminus \mathrm{Im}(\pi)$ $\rightsquigarrow$ repeat the process. We get some $f: X \to \mathbb{P}^r$ after finitely many steps (by composing all projection) which is surjective & w/ finite fibers.

<u>Lemma</u>: Let X be an irred topological space of finite dimension.

(i) If $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_m = X$ is a maximal chain $\Rightarrow \dim X_i = i$

(ii) If $Y \subsetneq X$ closed and irreducible $\Rightarrow \dim Y < \dim X$

(iii) X, Y projective varieties w/ the Zariski topology if $f: X \twoheadrightarrow Y$ surjective morphism $\Rightarrow$ every longest chain $Y_0 \subsetneq \cdots \subsetneq Y_n$ lifts to a chain in X.

<u>Lemma</u>: $X \subsetneq \mathbb{P}^n$ projective variety s.t. $[0:\cdots:0:1] \notin X$ (always possible via change of coordinates). Then $\forall f \in \mathbb{K}[x_0:\cdots:x_n]$ $\exists D >$ and $a_i \in \mathbb{K}[x_0:\cdots:x_{n-1}]$ homogeneous s.t.

$$f^D + f^{D-1}a_1 + \cdots + a_D = 0$$

<u>Corollary</u>: $X \subsetneq \mathbb{P}^n$ $\quad X \xrightarrow{\pi} \mathbb{P}^{n-1}$. If $Y \subset X$ subvariety s.t. $\pi(Y) = \pi(X) \Rightarrow Y = X$.

Proof if $Y \subsetneq X \Rightarrow \exists f \in I(Y) \setminus I(X)$

Let D be as in the preceeding lemma, and let it be minimal. Then

$$f^D + a_1 f_1^{D-1} + \dots + a_D = 0$$

$\Rightarrow a_D \in \langle f \rangle \subset I(Y)$. Not only that, a_D is a function on $\mathbb{P}^{n-1}$ vanishing on the image $\pi(Y)$, but not on $\pi(X)$. Contradiction.

Corollary : $\dim(X) = \dim \pi(X)$

Proof : lift $Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subseteq \pi(X)$ maximal chain

(if $Y_n \neq \pi(X) \Rightarrow$ it is an irred. component of it)

This means $\dim X \geq \dim \pi(X)$.

On the other hand $X_0 \subsetneq \dots \subsetneq X_n = X$ produces a chain of closed irred

$$Y_0 \subsetneq \dots \subsetneq Y_n = \pi(X)$$

$\neq \rightarrow$ previous lemma
closed by completeness

So $\dim \pi(X) \geq \dim X$ ∎

Coro : $\dim \mathbb{P}^n = n$

Proof $\mathbb{P}^0 \subsetneq \mathbb{P}^1 \subsetneq \dots \subsetneq \mathbb{P}^n$

Since they are irreducible

$$\dim \mathbb{P}^i < \dim \mathbb{P}^{i+1}$$

Now, since every projective variety surjects onto some $\mathbb{P}^n$ $\Rightarrow$ any $d = \dim X$ is the dimension of some projective space.

But any $i \geq 0$ is the dimension of some variety (by $\dim X_i = i$ in a longest chain and $\mathbb{P}^0 < \dots < \mathbb{P}^n$)

$\boxed{}$

<u>Prop</u>: If $\mathcal{U} \subset X$ open $\Rightarrow \dim \mathcal{U} = \dim X$

<u>Proof</u>: $\leq$) $\mathcal{U}_0 \subsetneq \dots \subsetneq \mathcal{U}_n = \mathcal{U}$ longest chain in $\mathcal{U}$ (topology induced from X) $\Rightarrow \overline{\mathcal{U}_0} \subsetneq \dots \subsetneq \overline{\mathcal{U}_n} = X$

$\geq$) a bit messy. Idea: $X_0 \subsetneq \dots \subsetneq X_n = X$ maximal chain $\rightsquigarrow X_0 \cap \mathcal{U} \subsetneq \dots \subsetneq X_n \cap \mathcal{U} = \mathcal{U}$

(Assume $X_0 \subset \mathcal{U}$)

Express $X_{i+1} = X_i \cup \overline{(X_{i+1} \cap X \setminus \mathcal{U})}$ if $\mathcal{U}_i = \mathcal{U}_{i+1}$.

$\square$

<u>Examples</u>

1) $\dim \mathbb{A}^n = \dim \mathbb{P}^n$ by the above.

2) $\mathbb{A}^{m+n} \subset \mathbb{P}^n \times \mathbb{P}^m$ open $\Rightarrow \dim \mathbb{P}^n \times \mathbb{P}^m = m+n$

3) $f \in \mathbb{K}[x_1 \dots x_n] \setminus \mathbb{K}$; $Z(f) \subset \mathbb{A}^n$ & $Z(f) \subset \mathbb{P}^{n-1}$ have dimension $n-1$

(Krull's Hauptidealsatz) + $Y \subset X$ closed $\Rightarrow \dim Y < \dim X$

codim $Z(f) \leq 1$

6 Analytic projective space vs. algebraic projective space.

Just as it happened for affine varieties, given that $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold (indeed, it consists of ~~n+1~~ open sets $\cong \mathbb{C}^n$ glued to one another by a holomorphic map on their intersection) we can take the holomorphic point of view.

Indeed $\underbrace{V(f_1, \dots, f_m)}_{\text{homogeneous}}$ can be assigned an analytic variety $V^{an}(f_1, \dots, f_m)$

In this case, the converse is also true:

<u>Thm</u> (Chow) Let $f_1 \dots f_m$ be holomorphic functions on $\mathbb{C}^{n+1}$, and let $V(f_1, \dots, f_m) \subset \mathbb{C}^{n+1}$ map to X under $\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n$. Then X is projective algebraic.

<u>Proof</u> (sketch)

- $V' = \pi^{-1}(X) \cup \{0\}$ is a cone, i.e. $t\bar{x} \in V'$ whenever $\bar{x} \in V'$
- Locally around 0, we can find holomorphic functions g_i s.t. $V'|_{U \ni 0} = V(g_1 \dots g_k)$ [Remmert, Stein]
- By reducing U $\rightsquigarrow$ $g_i(z) = \sum_{j=0}^{\infty} p_{i,j}(z)$ $\quad p_{i,j}$ homogeneous of degree j on the whole of U.

$$g_i(tz) = \sum t^j p_{i,j}(z) . \quad \text{Fix } z = z_0 \rightsquigarrow g_i(tz_0)$$

is a power series in one variable vanishing on an open set $(|t| < 1)$ $\Rightarrow$ it vanishes on $\mathbb{C}$

* Hence $\varphi_{ij} = 0$ on $\mathcal{U}$ $\rightsquigarrow$ also on V.

<u>Rks</u> 1) The proof is much more general and works in the setting of analytic subspaces of $\mathbb{P}^n$. These are closed subspaces locally defined by the vanishing of holomorphic sections.

The key point is that the [Remmert-Stein] theorem also applies to this context.

That is, $V \subset \mathbb{P}^n$ analytic subspace

$\pi^{-1}(V) \cup \{0\}$ is analytic subspace of $\mathbb{C}^{n+1}$ $\Rightarrow$ locally defined by vanishing of holomorphic sections.

2) From the proof we deduce that analytic cones in $\mathbb{C}^n$ are algebraic.