Algebraic geometry for physicists

PART I: from varieties to schemes

1. Affine varieties
2. Projective varieties
3. Schemes. Schemes versus analytic spaces.

PART II

4. Sheaves
5. Intersection theory.

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AFFINE VARIETIES

(I) Reminder on commutative algebra

$\mathbb{K}$ field (for us ; $\mathbb{R}$ or $\mathbb{C}$)

1). Algebras

def A $\mathbb{K}$-algebra A is a ring $(A, +, \cdot)$ (with $1_A \in A$) together with an action

$$\begin{aligned} \mathbb{K} \times A &\longrightarrow A \\ (\lambda, a) &\longmapsto \lambda \cdot a \end{aligned}$$

satisfying $\forall\ \lambda, \mu \in \mathbb{K}$, $a, b \in A$

$$\left.\begin{array}{ll} (M1) & \lambda(a+b) = \lambda\cdot a + \lambda\cdot b \\ (M2) & (\lambda+\mu)\cdot a = \lambda\cdot a + \mu\cdot a \\ (M3) & (\lambda\mu)\cdot a = \lambda\cdot(\mu\cdot a) \\ (M4) & 1\cdot a = a \end{array}\right\} \text{A is a } \mathbb{K}\text{-module (equiv. } \mathbb{K} \text{ vs)}$$

(A) $(\lambda a)(\mu b) = \lambda\mu \cdot ab$

Rk assuming $1_A \in A$ we identify $\mathbb{K}$ with $1\cdot\mathbb{K} \subset A$

Examples

- $\mathbb{C}$ is an $\mathbb{R}$-algebra
- $\mathbb{H}$ is an $\mathbb{R}$-algebra
- $\mathbb{C}[x_1, \dots x_n]$ (poly's in n-vars)

Rk the same def. applies if instead of $\mathbb{K}$ we take a ring B (B-algebra). Note that if A is a B-algebra, there exists

$$\begin{aligned} \varphi : B &\longrightarrow A \\ b &\longmapsto b\cdot 1_A \end{aligned}$$

Likewise $\varphi : B \longrightarrow A$ homomorphism induces a B-alg. structure on A by $b \cdot a = \varphi(b)a$.

Def Let A be a B-algebra. We say A is of finite type over B if $\exists\ a_1, \dots, a_n \in A$ such that $A = B[a_1, \dots, a_n]$. If $\varphi: B \to A$ is the correspon-ding morphism, φ is of finite type iff A is of f.t. $/B$.

WARNING! finite type $\Leftrightarrow$ ftely grated as an algebra
We say φ is finite if A is a ftely grated B-module

Examples

- $\mathbb{C}[x_1, \dots, x_n]$ is a $\mathbb{C}$-alg. of finite type
- $\mathbb{C}$ is a finitely grated $\mathbb{R}$-module

1.2) Rings and ideals. Localization.

Def: R comm. ring, $I \subseteq R$ ideal is a subring such that $IR = RI = I$. In other words: I is an R algebra contained in R.

Examples: • $x \cdot \mathbb{C}[x] \subset \mathbb{C}[x]$ is an ideal.

• $\mathbb{Z}$ has as ideals the subsets

$$\langle n \rangle = \{\text{numbers divisible by } n\}$$

Note that $\langle p \rangle$ for p prime satisfies that if $x \cdot y \in \langle p \rangle$ then either $x \in \langle p \rangle$ or $y \in \langle p \rangle$

Def: $I \subset R$ ideal satisfying that $\forall x, y \in R$ $xy \in I \Rightarrow x \in R$ or $y \in R$ is called prime ideal

Examples

- $I = x^2 \cdot \mathbb{C}[x]$ is not prime as $x^2 \in I$ but $x \notin I$
- $I = x \cdot \mathbb{C}[x]$ is prime, as the polynomial x is irreducible.

Exercise: prove that $f(x) \cdot \mathbb{C}[x]$ is prime iff $f(x)$ is irreducible.

Rk: If $I \subset R$ ideal, then $R/_I$ is a ring.

Examples: • $\frac{\mathbb{C}[x]}{\langle x^2 \rangle} \cong \mathbb{C} \oplus x \cdot \mathbb{C}[x]$, where $\langle f(x) \rangle = f(x) \cdot \mathbb{C}[x]$

• $\frac{\mathbb{C}[x]}{\langle x \rangle} \cong \mathbb{C}$

Def: a ring R is called

1) An integral domain if $x \cdot y = 0 \Rightarrow x = 0$ or $y = 0$ (i.e., if R has no divisors of 0)
2) Integrally closed if it is integral domain and letting $Fr(R) = \{ \frac{r}{s} \quad r, s \in R \}$, $p(x) = x^k + LOT \in R[x]$, if $p(x_0) = 0$, $x_0 \in Fr(R)$ then $x_0 \in R$.

Examples

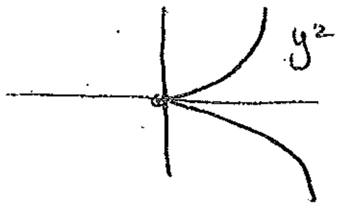
• $\frac{\mathbb{C}[x]}{\langle x^2 \rangle}$ is not an integral domain, as $x \cdot x = 0$

• $R/_I$ integral domain $\Leftrightarrow$ I is prime (Exercise)

• $\frac{\mathbb{C}[x,y]}{\langle y^2 - x^3 \rangle} \cong \mathbb{C}[t^2, t^3]$ is not integrally closed

Indeed $\overline{\mathbb{C}[t^2, t^3]}^{int} = \mathbb{C}[t] \subset \mathbb{C}(t) = \{ \frac{p(t)}{q(t)} \quad p, q \in \mathbb{C}[t] \}$

Related to singularity



$y^2 = x^3$.

Maximal ideals

Def An ideal $I \subsetneq R$ is maximal if $\forall J \subsetneq R$ ideal $I \not\subset J$.

Example $\langle x \rangle \subset \mathbb{C}[x]$ is maximal. Indeed, if $\langle x \rangle \subsetneq J$, then $\exists f(x) \in J \setminus \langle x \rangle \Leftrightarrow f(x) = c_0 + x \cdot g(x)$

Since $f(x) \in J$, $\langle x \rangle \subset J \Rightarrow c_0 \in J \Leftrightarrow J = \mathbb{C}[x]$.

~~Prop~~ (Krull) $(R, +, \cdot)$ comm. ring w/ 1_R. Then $\exists\, m \subset R$ maximal

Prop: • maximal ideals are prime
• every ideal is contained in a maximal one.
• m maximal $\Leftrightarrow$ R/m is a field
(Exercise) (use that a field is a ring s.t. all non-zero elements have an inverse)

Localization

$S \subset R$ multiplicatively closed subset. $1 \in S$. We define R_S the localization of R at S to be the ring

$$\frac{R \times S}{\sim} \quad \text{where } (a, s) \sim (b, t) \Leftrightarrow \exists\, u \in S \quad u(at - sb) = 0$$

Exercise: prove $\sim$ is an equivalence relation and R_S is a ring with $[(a,s)]_\sim + [(b,t)]_\sim = [(at + bs, st)]_\sim$

$$[(a,s)]_\sim \cdot [(b,t)]_\sim = [(ab, ts)]_\sim$$

NOT
We identify $[(a,s)]_\sim$ with $\frac{a}{s}$

Examples • $f(x) \in \mathbb{C}[x]$ $\quad \mathbb{C}[x]_f = \mathbb{C}[x]_{\langle f \rangle} = \left\{ \frac{g}{f^n} \right\}$

• If $\mathfrak{p} \subset R$ prime $\Rightarrow R \setminus \mathfrak{p}$ is multiplicatively closed. We denote $R_\mathfrak{p} = \left\{ \frac{r}{s} : s \notin \mathfrak{p} \right\}$ the localization of R at $\mathfrak{p}$.

Def a ring R is local if it contains a unique maximal ideal m.

Examples

- $\mathbb{C}, \mathbb{R}, K$ field (0 is the unique ideal $\neq K$)
- R_p $p \subset R$ prime. Indeed $\{ I \subset R_p \text{ ideals} \}$

(Exercise) $\longrightarrow \updownarrow 1-1$

$\{ I \subset R \text{ ideal} : p \supseteq I \}$

- $(z_1, \dots z_n) \in \mathbb{C}^n$

$m = \{ f \in \mathbb{C}[X_1 \dots X_n] : f(z_1, \dots z_n) = 0 \}$ is maximal.

$\mathbb{C}[X_1 \dots X_n]_m$ local

Def: R is normal if $\forall p \subset R$ prime, R_p is an integrally closed domain.

2) Affine varieties

2.1) As point sets

Def: $\mathbb{A}^n_{\mathbb{K}} = \{(\alpha_1, \dots, \alpha_n), \alpha_i \in \mathbb{K}\}$ the affine n-space over $\mathbb{K}$.

Def: $S \subset \mathbb{K}[x_1 \cdots x_n]$. We define

$$V(S) = \{\bar{\alpha} \in \mathbb{A}^n_{\mathbb{K}} : f(\bar{\alpha}) = 0 \ \forall f \in S\}$$

the affine ~~set~~ algebraic associated to S.

Examples

1) $\mathbb{A}^n = V(0)$
2) $\emptyset = V(1)$
3) $\{(\alpha_1, \dots, \alpha_n)\} = V(x_1 - \alpha_1, \dots, x_n - \alpha_n)$
4) $V(S) \cap V(S') = V(S \cup S')$
5) $V(S) \cup V(S') = V(S \cdot S')$

Remarks

1) Examples 1), 2), 4) and 5) above imply that affine sets are the closed set of a topology on $\mathbb{A}^n_{\mathbb{K}}$ (the Zariski topology).

> **Recall** A topology $\mathcal{T}$ on a set X is a family of subsets of X s.t. $\emptyset, X \in \mathcal{T}$, $X_i \in \mathcal{T} \Rightarrow \bigcup_{i \in I} X_i \in \mathcal{T}$ and $X_1, \dots, X_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^n X_i \in \mathcal{T}$ (open subsets).
> Equivalently, we may fix the closed subsets by choosing a family containing $\emptyset$, X, finite unions and arbitrary intersections.

2) Let $f, g \in S$. Then $V(S) \subset V(fa + gb)$ $\forall a, b \in \mathbb{K}[x$

$$\langle S \rangle := \{f_1 h_1 + \dots + f_m h_m : m \in \mathbb{N}, f_i \in S \quad h_i \in \mathbb{K}[x_1 \cdots x_n]\}$$

is the ideal generated by S. We have $V(S) = V(\langle S \rangle)$

(Exercise)

<u>Example</u>: affine sets in $\mathbb{A}^1_{\mathbb{K}}$

* By remark 2) $\rightsquigarrow$ they are $V(I)$ $I \subset \mathbb{K}[X]$ ideal
* Let $a, b \in \mathbb{K}[X]$ $\deg a \geq \deg b \Rightarrow \exists\, q, r \in \mathbb{K}[X]$ $\deg r < \deg b$

  $a = bq + r$

* If $a, b \in I \Rightarrow r \in I \Rightarrow$ it makes sense to choose $b \in I$ s.t. $\deg(f) \geq \deg(b)$ $\forall f \in I$.

  Then $\forall a \in I$ we have $a \in \langle b \rangle \Rightarrow I \subset \langle b \rangle$

  The converse is trivial as $b \in I$.

* <u>So</u> affine sets are zeroes of polynomials.

<u>Thm</u> (Hilbert's basis thm) every ideal in $\mathbb{K}[x_1 \dots x_n]$ is finitely generated

<u>Rk</u>: that's not quite the theorem ... It works when substituting $\mathbb{K}$ by any Noetherian ring R (i.e., a ring with ftely grated ideals.)

<u>Corollary</u>: algebraic sets of $\mathbb{A}^n_{\mathbb{K}}$ are zeroes of finitely many polynomials.

We have a dictionary.

Algebraic objects	$\longleftrightarrow$	Geometric objects
ideal I	$\longmapsto$	$V(I)$ alg. set
$I(X)$	$\longleftarrow$	$X \subset \mathbb{A}^n_{\mathbb{K}}$

where

$$I(X) = \{ f \in \mathbb{K}[x_1, \dots, x_n] \;:\; f(\bar{x}) = 0 \;\; \forall \bar{x} \in X \}$$

The correspondence is <u>not</u> bijective

Examples

1) $\langle x^2+1 \rangle \subset \mathbb{R}$ $\quad V(I) = \emptyset = V(1)$

2) $V(I(X)) = \overline{X}^{Z}$ the Zariski closure of X

3) $V((x-a_1)^{m_1}, \dots, (x-a_n)^{m_n}) = V(x-a_1, \dots, x-a_n) \subset \mathbb{C}^n$

Theorem (Hilbert's Nullstellensatz) $\mathbb{K}=\mathbb{C}$

$$I(V(J)) = \sqrt{J} = \{ f \in \mathbb{K}[x_1 - x_n] : f^n \in J \}$$

Rk: if $\mathbb{K}$ not algebraically closed $\Rightarrow$ false
(e.g. the ideal $\langle x^2+1 \rangle$ is prime ($\Rightarrow$ radical) but it is not $I(X)$ for any proper $X \subset \mathbb{R}$.)

Properties of $V(\cdot)$, $I(\cdot)$

$S, S' \subset \mathbb{K}[x_1 \dots x_n]$, $X, X' \subset \mathbb{A}^n_{\mathbb{K}}$

1) $X \subset X' \Rightarrow I(X) \supset I(X')$

2) $S \subset S' \Rightarrow V(S) \supset V(S')$

3) $X \subseteq V(I(X))$, $S \subseteq I(V(S))$ $\mathbb{K}=\mathbb{C}$, with equality if X is an algebraic set and S a radical ideal

4) $I(V(I(X))) = I(X)$ $\qquad V(I(V(S))) = V(S)$
(i.e. $I(X)$ is radical)

5) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$ $\qquad$ 6) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$

Coordinate rings and regular maps

Idea: the first way to study the geometry of a space is to look at the functions it admits. For example, polynomial functions on $\mathbb{A}^n_{\mathbb{C}}$ are given by $\mathbb{C}[x_1 \dots x_n]$. The latter ring also provides coordinates for the space which measure the degrees of freedom.

Examples: the plane $\{z=0\}$ inside $\mathbb{A}^2_{\mathbb{C}}$ has coordinate ring $\dfrac{\mathbb{C}[x,y,z]}{\langle z \rangle} \cong \mathbb{C}[x,y]$

- The hyperbola $\{xy-1=0\} \subset \mathbb{R}^2$ has coordinate ring $\frac{\mathbb{R}[x,y]}{\langle xy-1\rangle}$, namely, polynomial functions are sums $f(x)+g(y)$.

- The cusp $\{y^2=x^3\} \subset \mathbb{R}^2$ has coordinate ring

$$\frac{\mathbb{R}[x,y]}{\langle y^2-x^3\rangle} \hookrightarrow \mathbb{R}[t].$$

$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

The image is thus $R=\mathbb{R}[t^2,t^3]$. Note that the polynomial $(s)=s^2-t^2$ has a root in $\mathbb{R}(t)$, but not in R. This means R is not integrally closed. Its integral closure is in fact $\mathbb{R}[t]$.

__Def__: Given $X \hookrightarrow \mathbb{A}^n_{\mathbb{K}}$ an affine algebraic set, its coordinate ring is $\mathbb{K}[X] := \frac{\mathbb{K}[x_1,\dots x_n]}{I(X)}$.

We likewise define $\mathbb{K}(X)$ the field of fractions of X to be the set $\{\frac{f}{g} \; f,g \in \mathbb{K}[X]\}$.

Once functions have been studied, we move on to maps between varieties.

__Def__ Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine sets. A map $f: X \longrightarrow Y$ is said to be regular if $\exists f_1,\dots,f_m \in \mathbb{K}[X]$ such that $f(x)=(f_1(x) \dots f_m(x))$

Rk: • $f \in \mathbb{K}[x]$ is a regular function of $\mathbb{A}^1_{\mathbb{K}}$
• The choice of the f_i is not unique

• A regular map $f: X \longrightarrow Y$ induces a ring ($\mathbb{K}$-algebra) homomorphism

$$f^{\#}: \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$$
$$p(y) \longmapsto f^{\#}p(y) = (p\circ f)(y)$$

In particular, $X \overset{i}{\hookrightarrow} \mathbb{A}^n_{\mathbb{K}}$ yields

$$\mathbb{K}[x_1, \dots, x_n] \xrightarrow{i^{\#}} \mathbb{K}[X] \to 0$$

which is surjective. The kernel is $I(X)$

This yields a functor (contravariant)

$$\left\{\text{Affine Algebraic sets}\right\} \longrightarrow \left\{\begin{array}{l}\text{Reduced } \mathbb{K}\text{-algebras}\\ \text{of finite type}\end{array}\right\}$$

$$X \longmapsto \mathbb{K}[X]$$

Rk we need $X \hookrightarrow \mathbb{A}^n_{\mathbb{K}}$ to define $\mathbb{K}[\mathbb{A}^n_{\mathbb{K}}] \twoheadrightarrow \mathbb{K}[X]$

• Conversely: if we have $\mathbb{K}[Y] \xrightarrow{\varphi} \mathbb{K}[X]$ a morphism of $\mathbb{K}$-alg. of finite type. Then we find $X \to Y$ regular.

WARNING! morphism of $\mathbb{K}$-alg. of finite type implies

$$\begin{array}{ccccccccc}
0 \to & I(X) & \to & \mathbb{K}[x_1 \dots x_n] & \to & \mathbb{K}[X] & \to 0 \\
& \uparrow \varphi' & & \uparrow \tilde{\varphi} & & \uparrow \varphi & \\
0 \to & I(Y) & \to & \mathbb{K}[y_1 \dots y_m] & \to & \mathbb{K}[Y] & \to 0
\end{array}$$

Why should I care??

1) Algebraic manipulations are easier to handle.
2) It shows considering varieties as embedded in affine space is <u>complicated</u>. What if we take $\varphi : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ $\mathbb{K}$-algebra homomorphism?
We will see this later.

<u>Examples</u>

1) $y^2 = x^3$ is parameterized by
$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (t^2, t^3) \end{aligned}$$
Equivalently
$$\begin{aligned} \frac{\mathbb{R}[x,y]}{\langle x^3 - y^2 \rangle} &\hookrightarrow \mathbb{R}[t] \\ x &\longmapsto t^2 \\ y &\longmapsto t^3 \end{aligned}$$
We see the curve is not isomorphic to the line, as its ring is not integrally closed.

2) $y = x^\ell$ parameterised by
$$F : t \longmapsto (t, t^\ell)$$
F is an isomorphism whose inverse is
$$x \longleftarrow (x, y)$$

<u>Prop</u>: Let $f : X \to Y$ be a regular map of affine sets. Then

(i) $\mathbb{K}[Y] \hookrightarrow \mathbb{K}[X]$ is injective $\Leftrightarrow$ $f(X)$ is dense in Y

(ii) $\mathbb{K}[Y] \twoheadrightarrow \mathbb{K}[X]$ is surjective $\Leftrightarrow$ $X \xhookrightarrow{f} Y$ embedding

Example $xy=1$ $\qquad \mathbb{R}^2 \xrightarrow{F} \mathbb{R}$ has dense ima[ge]

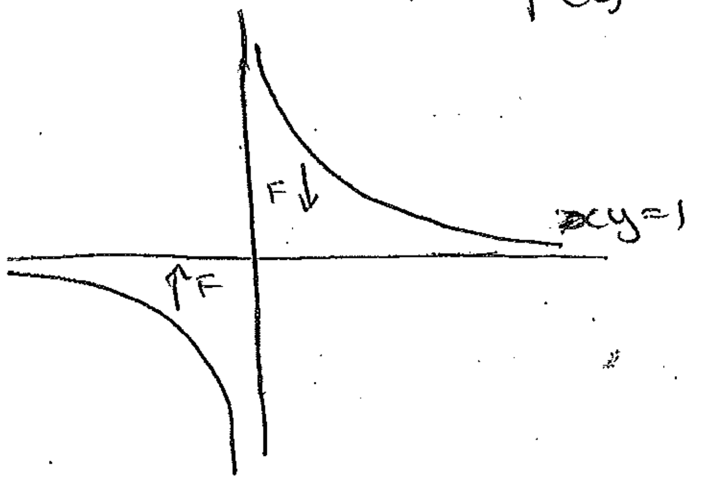
$\qquad\qquad\qquad\qquad (x,y) \longmapsto \ast$

The morphism

$$\mathbb{R}[t] \longrightarrow \frac{\mathbb{R}[x,y]}{\langle xy-1\rangle} \cong \mathbb{R}[x, \tfrac{1}{x}]$$

$$p(t) \longmapsto p(\ast)$$

is thus injective



<u>Prop</u>: $Y \cong X$ affine sets $\Leftrightarrow$ $\mathbb{K}[X] \cong \mathbb{K}[Y]$

<u>Proof</u> Exercise. Use that $f: X \to Y$ must be polynomial when we see $X \subset \mathbb{K}^n$ $Y \subset \mathbb{K}^m$.

<u>The Zariski topology</u>

On $\mathbb{A}^n_{\mathbb{R}/\mathbb{C}}$ we have two topologies: the usual topology, whose open sets are unions of balls, and the Zariski topology, whose open sets are complements of affine algebraic sets.

<u>Example</u>: The Zariski topology is much coarser than the usual topology. In $\mathbb{R}$ or $\mathbb{C}$ Zariski open sets are complements of finite sets. These are open in the usual.

Likewise, affine subsets Y of an affine algebraic set X are the closed sets of a topology in X.

<u>Properties</u> of the Zariski topology

1) Basic open sets are

$$D_f = \{ \bar{x} \in X : f(\bar{x}) \neq 0 \} \qquad f \in \mathbb{K}[X]$$

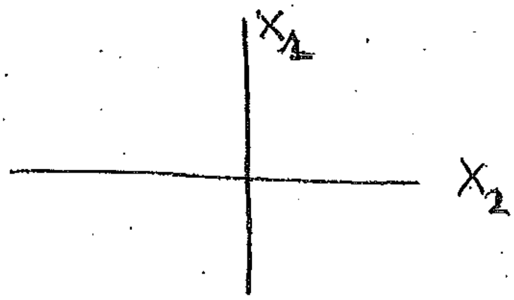
These play the role of open balls in the usual topology as the building blocks of open sets

2) Open sets are dense in irreducible algebraic sets (varieties). $U \subset X$ open, X irreducible $\Rightarrow \overline{U}^{z} = X$
* <u>add example</u>

3) $X \hookrightarrow \mathbb{A}^n_{\mathbb{K}}$. The Zariski topology of X is the same as the induced topology from the Zariski topology in $\mathbb{A}^n_{\mathbb{K}}$.

4) <u>Irreducible</u> components

$$X = \{ x \cdot y = 0 \} \subset \mathbb{R}^2$$
$$\parallel$$
$$\underset{\overset{\parallel}{X_1}}{V(x)} \cup \underset{\overset{\parallel}{X_2}}{V(y)}$$



In terms of coordinate ring:

$$\mathbb{K}[X] = \frac{\mathbb{K}[x,y]}{x \cdot y}$$

Note that $\bar{x} \neq 0 \neq \bar{y}$ but $\overline{xy} = 0$.

<u>Prop</u>: X is irreducible $\Leftrightarrow$ $\mathbb{K}[X]$ is an integral domain (see AV 2)

<u>Def</u>
Irreducible affine algebraic sets are called <u>affine varieties</u>

<u>Prop</u>': X is irreducible $\Leftrightarrow$ $\mathbb{I}(X)$ is a prime ideal.

Def: X topological space is said to be Noetherian if it has the descending chain condition (DCC) on closed subsets: every chain of closed subsets

$$X \supseteq X_1 \supseteq X_2 \quad \cdots$$

is stationary (i.e. $\exists\, n_0$ s.t. $\forall\ m > n_0 \quad X_m = X_{n_0}$)

Algebraically X alg. set is Noetherian for Zariski top. $\Leftrightarrow$ every ascending chain of ideals in $\mathbb{K}[X]$ is stationary $\underset{\text{Def}}{\Leftrightarrow}$ $\mathbb{K}[X]$ is a Noetherian ring

Def: X irreducible topological set. Then

$$\dim_{top}(X) = \sup_n \{\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X : X_i \text{ clos irred}\}$$

In the case X is an affine variety:

$$\dim_{Zar} X = \sup_n \{I(X) = \mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_n \neq 1\} \quad \mathfrak{P}_i \text{ prime ideals}$$

(Hilbert's basis thm)

Rk: $\left\{\begin{matrix}\text{prime ideals} \\ \text{in } \mathbb{K}[X]\end{matrix}\right\} \underset{X \text{ irred}}{\overset{1-1}{\longleftrightarrow}} \left\{\begin{matrix}\text{prime ideals in } \mathbb{K}[\mathbb{A}^n] \\ \text{containing } I(X)\end{matrix}\right\}$

Prop: X Noetherian topological space $\Rightarrow$
$X = X_1 \uplus \cdots \uplus X_n \quad X_i$ irreducible closed subsets.

Def the dimension of an affine alg. set X is $\dim X = \max_i \{\dim X_i : X_i \text{ irred component}\}$

Algebraic counterpart for varieties: $I(X) = \bigcap_{i=1}^{n} I_i \quad I_i$ prime
This is an example of primary decomposition

Examples

1) $\mathbb{A}^1_{\mathbb{K}}$ has dimension 1. Indeed $\mathbb{K}[x]$ is a principal ideal domain (namely, all its ideals are generated by one polynomial). So assume $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1$ are prime ideals. Then $\mathfrak{P}_0 = \langle f_0 \rangle$, $\mathfrak{P}_1 = \langle f_1 \rangle$ with $f_1 | f_0$. Hence, it must be $\mathfrak{P}_0 = 0$, as $\mathfrak{P}_0$ is prime $\Leftrightarrow$ it is 0 or grated by an irreducible polynomial.

The same argument implies $\mathfrak{P}_1$ is maximal.

2) $\dim \mathbb{A}^n_{\mathbb{K}} = n$. The inequality $\geq$ is easy:

$$\mathbb{A}^0 \subsetneq \mathbb{A}^1 \subsetneq \mathbb{A}^2 \subsetneq \cdots \subsetneq \mathbb{A}^n$$

For $\leq$, we need extra machinery...

8.2 Spectrum of a ring

We have seen that $\mathbb{K}[X]$ contains more information than X itself. Indeed x^2+1 has no zeroes in $\mathbb{R}$, so it defines the same variety as $\{1=0\}$. Nonetheless, inside $\mathbb{C}$ we do find solutions. This information is lost if we just consider $X \subset \mathbb{R} \subset \mathbb{C}$.

This difference between both empty sets is encoded in the spectra of the corresponding coordinate rings.

Déf: Let R be a ring. We define its spectrum

$$\operatorname{Spec}(R) = \{\mathfrak{P} \subsetneq R \text{ prime ideal}\}$$

Example

$$\mathrm{Spec}\left(\frac{\mathbb{R}[x]}{\langle x^2+1\rangle}\right) = \{0\}$$

$$\mathrm{Spec}\left(\frac{\mathbb{R}[x]}{\langle 1\rangle}\right) = \emptyset \quad \text{as} \quad \langle 0\rangle = \frac{\mathbb{R}[x]}{\langle 1\rangle}$$

Rk: the spectrum of $\mathbb{K}[X]$ parameterizes all irreducible subvarieties of X.

Examples: (1) $\mathbb{A}^1_{\mathbb{K}} = \mathrm{Spec}(\mathbb{K}[x]) = \{0, \langle \text{irred. polynomials}\rangle\}$

* 0 is the generic point, as its closure in the Zariski topology is the whole space.
* The remaining points are closed. Note that if $\mathbb{K}$ is alg. closed, then the only irreducible polynomials are degree one polynomials. The ideal $\langle ax+b\rangle = \langle x+\frac{b}{a}\rangle$. Monic such polys are in 1-1 correspondence with points of the line (and maximal ideals)

$$\langle x-a\rangle \longmapsto a$$

(2) $\mathbb{A}^0 = \mathrm{Spec}(\mathbb{K}[x]) = \{0\}$ point

The Zariski topology revisited

* Basic open sets: $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) : f \notin \mathfrak{p}\}$ $f \in R$
* Closed sets: $V(\mathfrak{a}) = \{\mathfrak{p} \supset \mathfrak{a}\}$ $\mathfrak{a} \subset R$ ideal

Properties: • $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$ $\Longrightarrow$ $\exists$ topology whose closed sets are $V(\mathfrak{a})$
• $V(\mathfrak{a}\cdot\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

Exercise: check the assertions in the above example.

Prove also that a point in $\mathrm{Spec}(R)$ is closed $\iff$ it is maximal.

Examples

① $V(\mathfrak{a}) = \mathrm{Spec}(R/\mathfrak{a})$

② $\mathrm{Spec}(R_{\mathfrak{p}}) = \{ \mathfrak{q} \in \mathrm{Spec}\, R \;:\; \mathfrak{q} \subseteq \mathfrak{p} \}$

③ The definition allows to consider more rings:

$$\mathrm{Spec}\left(\frac{\mathbb{C}[x]}{\langle x^2 \rangle} \right) = \{ \langle x \rangle \} \text{ is the fat point.}$$

Note that $\frac{\mathbb{C}[x]}{\langle x^2 \rangle} \cong \mathbb{C} \oplus \mathbb{C} \cdot x$ = degree 1 polys.

These are interpreted as degree 1 expansions of functions at $\{\langle x \rangle\}$, namely values + 1st derivative. So $\mathrm{Spec}\left(\frac{\mathbb{C}[x]}{\langle x^2 \rangle} \right)$ is a germ of a line, i.e, an infinitesimal line.

A way to think of $\frac{\mathbb{C}[x]}{\langle x^2 \rangle}$ is as defining an infinitesimally small parameter ε, so small that it squares to 0. This allows us to define tangent space without having resource to differential analysis. We call the above ring the ring of dual numbers, and denote it by $\mathbb{C}[\varepsilon]$.

Properties of the spectral topology

- $\mathrm{Spec}(R)$ is compact.
- $\mathrm{Spec}(R)$ is irreducible $\iff$ $\exists$ generic point.

Relation with the previous definition

How should one think about $\mathrm{Spec}(R)$? When $R = \frac{\mathbb{C}[y_1 \cdots y_n]}{I}$ we have

$$\{ x \in V(I) \} \xleftrightarrow{1-1} \{ m \in \mathrm{Spec}_{max}(R) \}$$

$$x \longmapsto m_x = \{ f \in R \;:\; f(x) = 0 \} = \langle y_1 - x_1, \ldots, y_n - x_n \rangle$$

$$(a_1, \ldots, a_n) \longmapsfrom \langle y_1 - a_1, \ldots, y_n - a_n \rangle$$

Functions on $V(I)$ ~~where~~ elements $f \in R$. In a similar way, $f \in R$ defines

$$f : \operatorname{Spec}_{max}(R) \longrightarrow \mathbb{C}$$
$$m \longmapsto \bar{f} \in R_m/m \cong \mathbb{C} \quad \text{(Nullstellensatz weak v.)}$$

To define this

$$\begin{array}{ccccc} R & \Longrightarrow & R_m & \longrightarrow & R_m/m \cong \mathbb{C} \\ f & \longmapsto & \frac{f}{1} & \longmapsto & \left[\frac{f}{1}\right]_m =: \bar{f} \end{array}$$

This extends to $\operatorname{Spec}(R)$ by allowing the evaluation set to vary. Namely

$$f(p) \in R_p/p$$

<u>WARNING</u>: a "function" is not defined by its values at all points of $\operatorname{spec}(R)$. For example, the 0 function corresponds to $f \in R$ s.t. $f \in \bigcap_{p \in \operatorname{Spec}} p$ which is the set of nilpotent elements (Exercise). So for example, in $\operatorname{Spec} \mathbb{C}[x]/x^2$ the 0 function is given by both 0 and x.

More reasons why I should care

① We saw that $f : X \to Y$ (affine alg. sets) yields a morphism of ~~f.g.~~ ~~type~~ reduced $\mathbb{K}$-algebras $f^{\#} : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$

These morphisms are dependent on the embeddings $X \hookrightarrow \mathbb{A}^n_{\mathbb{K}}$, $Y \hookrightarrow \mathbb{A}^m_{\mathbb{K}}$. Is there an intrinsic way to define this?

Note that every ring homomorphism

$$\varphi : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$$

induces

$$\varphi^{-1} : \operatorname{Spec}(\mathbb{K}[X]) \longrightarrow \operatorname{Spec}(\mathbb{K}[Y])$$
$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$$

<u>Problem</u> $\underset{\substack{\uparrow \\ \text{maximal}}}{\varphi^{-1}(\mathfrak{m})}$ need not be maximal

e.g. $\mathbb{K}[x] \hookrightarrow \mathbb{K}(x)$

$$0 \longmapsto 0 \in \operatorname{Spec}_{\max}(\mathbb{K}(x))$$

<u>Rk</u>: in some rings (pm rings) every prime ideal is contained in a unique maximal one, so we can recover the geometric intuition by assigning

$$\mathfrak{m} \longmapsto \mathfrak{m}_{\varphi^{-1}(\mathfrak{m})} = \text{unique maximal containing } \varphi^{-1}(\mathfrak{m})$$

Why would we want to do this? See next section...

(2) The tangent space as an affine variety

$X \subset \mathbb{C}^n$ alg. affine set, let $I(X) = \langle f_1, \dots, f_k \rangle$

$$TX = \{(x, v) \quad x \in X, v \in T_x X\}$$

<u>Lemma</u>: $\operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[X], \mathbb{C}[\varepsilon]) \Longleftrightarrow TX$

<u>Prop</u>: $TX \subset T\mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n = \operatorname{Hom}(\mathbb{C}[x_1, \dots, x_n], \mathbb{C}[\varepsilon])$ is an algebraic set whose ideal consists of $\varphi \in \operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_n], \mathbb{C}[\varepsilon])$

$\varphi(I) = 0$

Idea of the proof

* $\varphi : \mathbb{C}[x_1, \dots x_n] \longrightarrow \mathbb{C}[\varepsilon] \rightsquigarrow (a_1, \dots, a_n; b_1, \dots, b_n)$

$x_i \mapsto a_i + b_i \varepsilon$

* $\mathrm{Hom}_{\mathbb{C}\text{alg}}(\mathbb{C}[X], \mathbb{C}[\varepsilon]) = \{\varphi : \mathbb{C}[x_1 \dots x_n] \to \mathbb{C}[\varepsilon] : \varphi(I) = 0\}$

$= \{\varphi : \varphi(f_j) = 0\} \qquad I = \langle f_1 \dots f_N \rangle$

* $f_j = \sum_I c_I x_1^{i_1} \dots x_n^{i_n} \rightsquigarrow \varphi(f_j) = r_j + s_j \varepsilon$

poly functions on a_i, b_i

* $\{\varphi(I) = 0\} \Leftrightarrow \{(\bar{a}, \bar{b}) \in \mathbb{C}^n \times \mathbb{C}^n : f_j(\bar{a}) = 0, \; s_j(\bar{a}, \bar{b}) = 0\}$

3) Affine varieties as ringed spaces

Regular functions

Geometry is concerned with local properties, as well as global. In particular, we may want to know functions locally defined at points.

Def: X irreducible $\mathbb{K}(X) = \{\frac{f}{g} : g, f \in \mathbb{K}[X]\}$ the field of rational functions. Let $x \in X$, $\varphi \in \mathbb{K}(X)$ is said to be regular at x if $\varphi = \frac{f}{g}$ locally around x $f, g \in \mathbb{K}[X]$ $g(x) \neq 0$.

Rk: • φ is regular at $x \Rightarrow$ it is regular on $\{g \neq 0\}$

• φ " " " " $\Leftrightarrow \varphi \in \mathbb{K}[X]_{m_x}$

Given $U \subset X$ open, we may assign to it

$$\mathcal{O}_X(U) = \{\varphi \in \mathbb{K}(X) : \varphi \text{ regular at } x \;\forall x \in U\}$$
$$= \bigcap_{x \in U} \underbrace{\mathcal{O}_{X,x}}_{\text{reg. functions at } x}$$

Theorem:

1) Let $f \in \mathbb{K}[X]$. Then $\mathcal{O}_X(D_f) = \mathbb{K}[X]_f$ ($\mathbb{K}$ alg. closed)

2) $\mathcal{O}_X(X) = \mathbb{K}[X]$ (use 1) for $f=1$)

3) Let $\mathcal{O}_{X,p} = \{\text{regular functions at } p\} = \mathbb{K}[X]_{m_p}$

Then $\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U)$, which can be interpreted as germs of rational functions at p. These are rational functions defined on a neighbourhood of p (say φ_V, φ_U) modulo the equivalence relation

$$\varphi_V \sim \varphi_U \iff \exists\, W \subset V \cap U \quad \varphi_V|_W = \varphi_U|_W$$

Example: beware, the fact that $\varphi \in \mathcal{O}_X(U)$ does not imply that $\varphi = \frac{f}{g}$ $f, g \in \mathbb{K}[X]$.

Let $X \subset \mathbb{A}^4$ be: $x_1 x_4 = x_2 x_3$ (*)

$$U = \{(x_1, \dots, x_4) : x_2 \neq 0 \text{ or } x_4 \neq 0\} = U_{x_2} \cup U_{x_4}$$

On $U_{x_2} \rightsquigarrow \frac{x_1}{x_2}$ is defined

$U_{x_4} \rightsquigarrow \frac{x_3}{x_4}$ " "

they match on $U_{x_2} \cap U_{x_4}$ by (*)

$\Downarrow$

they define a regular function φ

But $\nexists\, f, g \in \mathbb{K}[X]$ s.t. $\varphi = \frac{f}{g}$ on U.

Rk due to ambiguity: $x_1 x_4 - x_3 x_2 \in I(X)$ not well defined. See next example

The above theorem implies that $\mathcal{O}_X : \{\text{open sets}\} \to \{\text{rings}\}$ satisfies

Pre-sheaf axioms

(PS1) $\forall V \subset \mathcal{U} \quad \exists \rho^{\mathcal{U}}_V \; \mathcal{O}_X(\mathcal{U}) \longrightarrow \mathcal{O}_X(V)$ (restriction)

(PS2) $\mathcal{O}_X(\emptyset) = 0$

(PS3) $\rho^{\mathcal{U}}_{\mathcal{U}} = id$

(PS4) $W \subset V \subset \mathcal{U} \Rightarrow$

$$\begin{tikzcd} \mathcal{O}_X(\mathcal{U}) \arrow[rr, "\rho^{\mathcal{U}}_V"] \arrow[dr, "\rho^{\mathcal{U}}_W"'] & \circlearrowleft & \mathcal{O}_X(V) \arrow[dl, "\rho^V_W"] \\ & \mathcal{O}_X(W) & \end{tikzcd}$$

(S1) $V_i \to \mathcal{U}$ open covering

If $s \in \mathcal{O}_X(\mathcal{U})$ satisfies $s|_{V_i} := \rho^{\mathcal{U}}_{V_i}(s) = 0 \; \forall i \Rightarrow s = 0$

(sections determined by their local restrictions)

(S2) $V_i \to \mathcal{U}$ open covering

If $s_i \in \mathcal{O}_X(V_i)$ are s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then $\exists\, s \in \mathcal{O}_X(\mathcal{U})$ s.t. $s|_{V_i} = s_i$

(Compatible local sections paste to global sections)

<u>NAMELY</u> $\mathcal{O}_X$ is a sheaf of rings on X

<u>Example</u> $\mathcal{U} = \mathbb{C}^2 \setminus \{0\}$. We have $\mathcal{O}_{\mathbb{C}^2}(\mathcal{U}) = \mathbb{C}[x,y]$

Indeed, let $\varphi \in \mathcal{O}_{\mathbb{C}^2}(\mathcal{U})$.

<u>Claim 1</u> $\varphi = \frac{f(x,y)}{g(x,y)}$ globally.

Let $p=(a,b)$, $(a',b')=p' \in \mathbb{C}^2 \setminus \{0\}$. Then $\varphi = \frac{f(x,y)}{g(x,y)}$ around p

$\varphi = \frac{f'}{g'}$ around p'.

Thus $fg' = f'g$ on $W = \{g \neq 0\} \cap \{g \neq 0\}$.

Now, $\overline{U} = \mathbb{C}^2$ $\quad \overline{W} = \mathbb{C}^2$

$\overline{W} = \cap V(I)$

$W \subset V(I)$

Since $W \subset \{fg' = f'g\} \Rightarrow \mathbb{C}^2 \subset \{fg' = f'g\} \Rightarrow fg' = f'g$ on the whole of $\mathbb{C}^2$

? Why isn't this contradictory with the example $X \subset \mathbb{A}^4$, $X = \{x_1x_4 = x_2x_3\}$?

<u>Claim 2</u> $g(x,y) = $ constant.

~~Intuition~~ $\nexists$ any poly vanishing just at 0 as any poly cuts a codim 1 subvariety.

By the theorem, $g(x,y)$ constant $\Leftrightarrow \frac{f}{g}$ is regular on $\mathbb{A}^2_{\mathbb{C}}$. Let's check that $g(x,y)$ cannot just vanish at 0. Let

$$g(x,y) = f_0(x) + f_1(x)y + \dots + f_n(x)y^n$$ & assume $g(x,y) = 0 \Leftrightarrow (x,y) = 0$.

<u>Step 1</u> $f_0(x) = x^m$ for some $m \neq 0$

<u>Step 2</u> $g(x,0) = x^m \neq 0$

$\exists y_0 \ll 1$: $g(x,y_0) \neq 0$. Also $g(0,y_0) \neq 0$ as otherwise $(0,0)$ would not be the only root. Similarly, if $g(x,y_0) \neq$ constant $\Rightarrow$ it has a root (which by assumption is not possible) $\Rightarrow g(x,y_0) = $ constant $\Rightarrow m = 0$

<u>Contradiction</u>

Exercise: redo proof as in former example using
$x_1 \neq 0$ $x_2 \neq 0$

Rk: the above example is the algebraic version of Hartog's extension theorem: a ~~holomorphic~~ function on $U \subset \mathbb{C}^n$ open which is holomorphic on $U \setminus K$, K compact, $U \setminus K$ connected, extends uniquely to U. A parallelism between the analytic and the algebraic world exists and will be made clear soon.

Now, the structure sheaf gives us the extra information we have been missing

Def: X affine alg. var., $\mathcal{O}_X$ structure sheaf.
$(X, \mathcal{O}_X)$ is a locally ringed space, meaning:

- A topological space X
- A sheaf of rings whose stalks (that is $\mathcal{O}_{X,x}$) are local rings.

Recall: a ring R is local if it has a unique maximal ideal. A way to produce local rings is via localization at prime ideals. Note that if S is any ring
- $\mathrm{Spec}(S_{\mathfrak{p}}) = \{\mathfrak{q} \in \mathrm{Spec}(S) : \mathfrak{q} \subseteq \mathfrak{p}\}$
- Similarly for ideals.
- $S_{\mathfrak{p}} / I S_{\mathfrak{p}} = \left(S / I \right)_{\mathfrak{p}}$ $I \subset S$ ideal, $\mathfrak{p} \in \mathrm{Spec}(S)$

Since $\mathcal{O}_{X,x} = \mathbb{K}[X]_{m_x} \rightsquigarrow \checkmark$

With this, we can now establish yet another correspondence between geometric and algebraic data:

$$\left\{\begin{array}{l}\text{Ringed affine} \\ \text{varieties } (X, \mathcal{O}_X)\end{array}\right\} \xleftrightarrow{\ 1-1\ } \left\{\begin{array}{c}\text{reduced} \\ \mathbb{K}\text{-alg } \mathbb{K}[X] \\ \text{of finite type}\end{array}\right\}$$

We still need to understand what morphisms mean on the LHS. They have to be continuous on topological spaces $f: X \to Y$, but they should respect the local data given by structure sheaves. So a morphism consists of $(f, f^{\#})$ $f: X \to Y$ continuous

$$f^{\#}_{U} : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$

such that $f^{\#}_V \circ \rho^U_V = \rho^{f^{-1}(U)}_{f^{-1}(V)} \circ f^{\#}_U$

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "f^{\#}_U"] \arrow[d, "\rho^U_V"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}(f^{-1}(U)) \arrow[d, "\rho^{f^{-1}(U)}_{f^{-1}(V)}"] \\ \mathcal{O}_Y(V) \arrow[r, "f^{\#}_V"'] & \mathcal{O}(f^{-1}(V)) \end{tikzcd}$$

Moreover, the images should still be maps to $\mathbb{K}$ $\Rightarrow f^{\#}_U$ is a $\mathbb{K}$-alg homomorphism.

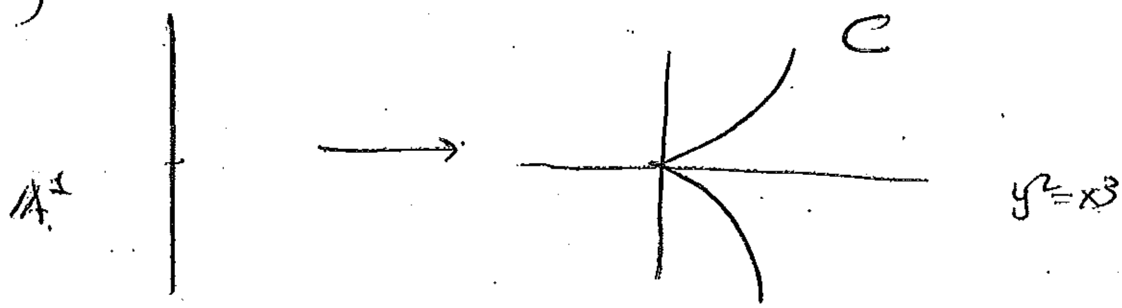
Rk given $f \rightsquigarrow f^{\#}$ pullback

What have we gained? The case of affine varieties is somewhat peculiar, as $\mathcal{O}_X$ is determined by global sections $\mathcal{O}_X(X) = \mathbb{K}[X]$. Nevertheless, think of $\mathbb{P}^n_{\mathbb{C}}$. Globally there are no holomorphic functions other than constants. But locally, $\mathbb{P}^n_{\mathbb{C}}$ is $\mathbb{A}^n_{\mathbb{C}}$; so there are plenty! Moreover, for spectra of rings, this is the correct approach. With the appropriate sheaves, these provide the first example of an affine scheme.

The following examples prove the right concept is that of a ringed space:

Examples

(1) $\{y^2 = x^3\} = C$



$$f : \mathbb{A}^1 \longrightarrow C \qquad t \longmapsto (t^2, t^3)$$

is a morphism which has an inverse

$$f^{-1} : C \longrightarrow \mathbb{A}^1$$

$$(x,y) \longmapsto \begin{cases} \dfrac{y}{x} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

But f^{-1} is not a morphism, as otherwise

$$k[t] \cong \frac{k[x,y]}{\langle x^3 - y^2 \rangle}$$ which is false.

(2) A priori, open sets of affine varieties are not affine varieties in the sense of $V(I)$, but they may be isomorphic to some. For example $\{f \neq 0\}$ is affine in this sense

$X \subset \mathbb{A}^n \qquad Y \subset X \times \mathbb{A}^1$

$\{(\underline{x}, t) : t f(x) - 1 = 0\}$

$\pi : Y \longrightarrow X$ is an iso onto $\{f \neq 0\}$

$(x, t) \mapsto x$

The structure sheaf is $\mathcal{O}_{X,f}$

$$\pi^{-1} : D_f \longrightarrow Y$$

$$x \mapsto \left(x, \frac{1}{f(x)}\right)$$

(3) $\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}$ is not an affine variety within $\mathbb{A}^2_{\mathbb{C}}$

If it where, $\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{X|\mathcal{U}}$, as $\exists\, D(f_i)_i \to \mathcal{U}$ covering.
However $\mathcal{O}_X|_{\mathcal{U}}(\mathcal{U}) = \mathcal{O}_X(X) = \mathbb{K}[x,y]$ by a preceding example.

But the correspondence would yield $(X, \mathcal{O}_X), (\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ are isom.

Hence $\mathcal{U}$ is not an affine variety. However
$\mathcal{U} = D_x \cup D_y$ which are affine.

<u>Moral</u>: we should be allowed to paste affine varieties
These are the first examples of schemes which are not affine.

(2.4) A word on affine schemes

For the sake of completeness, although this is not the main scope of this section, let me say something else about spectra of rings.

The spectrum $X = \mathrm{Spec}(R)$ can be endowed with a structure sheaf as follows:

* For each $\mathfrak{p} \in X$ define $\mathcal{O}_{X,\mathfrak{p}} = R_{\mathfrak{p}}$.
* For $\mathcal{U} \subset X$ open, let

$$\mathcal{O}_X(\mathcal{U}) = \left\{ s : \mathcal{U} \longrightarrow \prod_{\mathfrak{p} \in \mathcal{U}} R_{\mathfrak{p}} : s(\mathfrak{p}) \in R_{\mathfrak{p}} \text{ and locally } s(\mathfrak{p}) = \frac{f}{g} \right\}$$

* Locally here means $\exists\, V \ni \mathfrak{p}$ open s.t. $\forall\, \mathfrak{q} \in V$
$s(\mathfrak{q}) = \dfrac{f}{g}$

Since $\frac{f}{g} \in R_{\mathfrak{q}}$ $\forall\, \mathfrak{q} \in V$ it must be $g \notin \mathfrak{q}$

Note that when $R = \mathbb{C}[X]$, this restricts on the maximal spectrum to the structure sheaf.

Def: an affine scheme is the ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec} R})$ for some R ring.

ō check $\mathcal{O}_{\mathrm{Spec} R}$ is a sheaf

Comparison with classical varieties

The goal is to match local ringed spaces and rings. For this, morphisms need to be defined from ring homomorphisms. This implies that the pullback construction is not enough, but we ~~need~~ to ask for $(f, f^{\#})$ to satisfy

$$f^{\#}\ \mathcal{O}_{X, f(p)} \longrightarrow \mathcal{O}_{Y, p} \quad \text{has} \quad f^{\#} m_{Y, f(p)} = m_{X, p}$$

This restriction yields an correspondence (actually, an equivalence of categories)

$$\{\text{Affine schemes}\} \longleftrightarrow \{\text{Rings}\}$$

* Our former affine algebraic sets ~~would~~ are, in particular, schemes over $\mathbb{K}$, in the sense that $\exists\ \mathrm{Spec}(\mathbb{K}[X]) \longrightarrow \mathrm{Spec}\,\mathbb{K}$ induced by the $\mathbb{K}$ algebra structure.

* If we consider morphisms over $\mathrm{Spec}\,\mathbb{K}$ (namely

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{K}[X]) \arrow[rr] \arrow[dr] & & \mathrm{Spec}(\mathbb{K}[Y]) \arrow[dl] \\ & \mathrm{Spec}\,\mathbb{K} & \end{tikzcd}$$)

then $\{\text{alg. sets}\} \cap$

$$\left\{\begin{matrix}\text{Affine schemes} \\ \text{over } \mathrm{Spec}(\mathbb{K})\end{matrix}\right\} \longleftrightarrow \{\mathbb{K} \text{ algebras}\}$$

Adding "finite type" and "reduced" on both sides we have what we need.

3) Comparison with the analytic category (cf. [Toy])

On $\mathbb{C}^n$ we can use analysis in complex variables to study subspaces. For instance, given $f_1 \cdots f_m$ holomorphic, we can consider

$$V^{an}(f_1, \dots, f_m) = \{ z \in \mathbb{C}^n \mid f_i(z) = 0 \}$$

These are called <u>analytic closed sets</u>. If $V^{an}(f_i)$ is irreducible, it is called analytic subvariety.

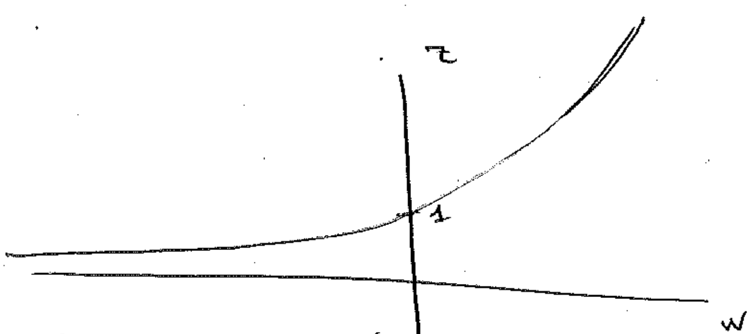
In particular, if $f_1 \cdots f_n \in \mathbb{C}[x_1, \dots x_n] \rightsquigarrow$ to $V(\langle f_1 \cdots f_n \rangle)$ we assign $V^{an}(\langle f_1, \dots, f_n \rangle)$.

<u>Rk</u>: Let $\mathcal{O}^{an}_{\mathbb{C}^n}(\mathbb{C}^n)$ be the ring of holomorphic functions on $\mathbb{C}^n$. It is a ring (actually, a $\mathbb{C}$-algebra) so it makes sense to consider ideals. We have that $V^{an}(f_1, \dots, f_n) = V^{an}(\langle f_1, \dots f_n \rangle)$ $\quad \langle f_1 \dots f_n \rangle \subset \mathcal{O}^{an}_{\mathbb{C}^n}(\mathbb{C}^n)$ ideal. This ideal is bigger than the one inside $\mathbb{C}[z_1, \dots z_n]$ (when $f_i \in \mathbb{C}[z]$), even when the sets of points are the same.

How about the converse? When is an analytic variety algebraic?

<u>Example</u>

$$V = \{ z = e^w \} \subset \mathbb{C}^2$$



$\rightarrow f(z,w) = z - e^w$ is not a polynomial

$\rightarrow$ BUT it can be approximated by such

$$f_N = z - \sum_{j=0}^{N} \frac{w^i}{i!}$$

$\rightarrow$ <u>Idea</u> $\quad I = \langle f_N : N \in \mathbb{N} \rangle \subset \mathbb{C}[z,w] \rightsquigarrow I = \langle f_1, \dots, f_M \rangle$

→ Problem: $f \notin I$, nor do $f_N|_V \equiv 0$. The line at ∞ would solve this

→ Compactify (more in next section).

Structure sheaf

$X \subset \mathbb{C}^n$ analytic set. Let $\mathcal{O}_X^{an}$ be the sheaf such that

- $\mathcal{O}_X^{an}(U) = \{\varphi \text{ holomorphic on } U\}$ U open
- $\mathcal{O}_{X,p}^{an} = \{\varphi_U \text{ hol on } U \ni p\} / \sim$

where $\varphi_U \sim \varphi_V \Leftrightarrow \exists W \subset U \cap V$ s.t. $\varphi_U|_W = \varphi_V|_W$.

(germs of functions)

This allows to see $(X, \mathcal{O}_X^{an})$ as a ringed space.

Rk: unlike polynomial functions, holomorphic functions are local in nature (just as regular functions are). So regular functions are more appropriate to compare to. On the other hand, this local nature of holomorphic functions makes it hard to pass from local to global information.

Properties of $\mathcal{O}_X^{an}$

a) $\mathcal{O}_{\mathbb{C}^n, p}^{an}$ is noetherian $\cong \mathbb{C}\{z_1, \dots z_n\}$ converging series

b) $\mathcal{O}_{X,p}^{an}$ is noetherian (follows from a))

c) $\mathcal{O}_{X,p}^{an}$ is local with maximal ideal $m_p = \{f \in \mathcal{O}_{X,p} : f(p) = 0\}$

d) $\mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{X,p}^{an}$ induces an isomorphism between their completions w.r.t. m_p

Recall: Let (R, m) be a local ring.

$$\hat{R} = \varprojlim_{\mathbb{N}} R/_{m^n}$$

e.g.

$$\widehat{\mathbb{C}\{z_1, \dots, z_n\}} = \varprojlim_{m \in \mathbb{N}} \frac{\mathbb{C}\{z_1, \dots, z_n\}}{\langle z_1, \dots, z_n \rangle^m} = \{ (f_m) : f_m \in R_{(m)} \}$$

$$\underset{R_{(m)}}{\|} \quad \bigoplus_{n=0}^{n-1} \mathbb{C}[z_1, \dots, z_n]_n$$

$f_m \mapsto f_{m+1}$

where $R_{(n)} \longrightarrow R_{(n-1)}$

$[f(\bar{z})] \longrightarrow [f(\bar{z})]$

$$\widehat{\mathbb{C}\{z_1 \dots z_n\}} = \mathbb{C}[[z_1, \dots, z_n]]$$

$$\widehat{\mathbb{C}[z_1, \dots, z_n]}_{\langle z_1 \dots z_n \rangle} = \varprojlim_{m} \frac{\mathbb{C}[z_1, \dots, z_n]}{\langle z_1, \dots, z_n \rangle^m} = \mathbb{C}[[z_1, \dots, z_n]]$$