RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 08.05.2013

Exercise sheet 3
Lie groups and hyperboloids
To hand in by Wednesday May 15, 13:00

Exercise 1. Consider the following subsets of the space of matrices:

$$O(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid Q^T Q = QQ^T = I\}$$

$$U(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{C}) \mid \bar{U}^T U = U\bar{U}^T = I\}$$

1. Prove that these subsets are groups for matrix multiplication.
2. Prove that they are submanifolds of $\mathrm{Mat}(n, n, \mathbb{R})$ or $\mathrm{Mat}(n, n, \mathbb{C})$, respectively, and compute their dimension.
3. Prove that they are compact subsets.
4. Show that some of them are connected and some are disconnected.
5. Prove that they are Lie groups.
6. Find the tangent space at the identity, more precisely prove that:

$$T_I O(n) = \mathfrak{o}(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid Q^T + Q = 0\}$$

$$T_I U(n) = \mathfrak{u}(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid \bar{U}^T + U = 0\}$$

7. Prove that the Lie bracket is given by the matrix commutator $[A, B] = AB - BA$.
8. Compute the Killing form of the two groups, and verify that it is positive definite.
9. Find an example of a symmetric space that is not simply connected.

Exercise 2. Consider the vector space $\mathbb{C}^{n+1}$ with the following Hermitian form of signature $(1, n)$:

$$h(x, y) = -x_0\bar{y}_0 + \sum_{i=1}^{n} x_i\bar{y}_i$$

The complex hyperbolic n-space $\mathbb{CH}^n$ is defined as:

$$\mathbb{CH}^n = \{v \in \mathbb{C}^{n+1} \mid h(v, v) = -1\}$$

Show that the tangent space at every point is the orthogonal space at that point for the form h, and show that the restriction of h to this tangent space defines a Riemannian metric on $\mathbb{CH}^n$ that makes it a symmetric space.

Exercise 3. Given an index $s \in \{0, \dots, n+1\}$, consider the bilinear form b on $\mathbb{R}^{n+1}$ of signature $(s, n+1-s)$ defined by

$$b(x, y) = -\sum_{i=0}^{s-1} x_0 y_0 + \sum_{i=s}^{n} x_i y_i$$

Define the semi-Riemannian sphere as

$$\mathbb{S}_s^n = \{v \in \mathbb{R}^{n+1} \mid b(v, v) = 1\}$$

and the semi-Riemannian hyperbolic space as

$$\mathbb{H}_{s-1}^n = \{v \in \mathbb{R}^{n+1} \mid b(v, v) = -1\}$$

Show that this space inherits a natural semi-Riemannian metric from b (in the usual way) and show that for this metric the geodesic symmetry at every point is an isometry. This is an example of a semi-Riemannian analog of symmetric spaces.