RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie I
Heidelberg, 22.01.2013

EXERCISE SHEET 13

Fundamental group and coverings

To hand in by January 29, 14:00

Exercise 1. Prove that a connected manifold is simply connected if and only if, for any $x \in M$, $\pi_1(M, x) = 0$.

Exercise 2. Let M be a connected manifold.

(a) Let $\{(U_i)\}_{i \in I}$ be an open cover of M, and $\gamma : [0, 1] \to M$ be a continuous map. Prove that there are a finite number of points $0 = a_0 < a_1 < \cdots < a_k = 1$ such that, for all j, $\gamma|_{[a_j, a_{j+1}]}$ is contained in one of the U_i.

(b) Prove that if $M = A_1 \cup A_2$, with A_1, A_2 connected and simply connected open subsets such that $A_1 \cap A_2$ is connected, then M is simply connected. [Hint: take any loop in M and show that it is null-homotopic.]

(c) Generalise in the following way: Prove that if $M = A_1 \cup \cdots \cup A_n$, with A_i connected and simply connected open subsets such that, for all i, j, $A_i \cap A_j$ is connected and $A_1 \cap \cdots \cap A_n \neq \emptyset$, then M is simply connected.

(d) Prove that the spheres S^n_1 for $n \geq 2$ and the complex projective spaces $\mathbb{CP}^n$ for $n \geq 1$ are simply connected.

Exercise 3. Let M, N be manifolds, and $p : M \to N$ be a covering, such that there exists a point $x \in M$ such that $p^{-1}(x)$ is a finite set with n points.

(a) Prove that for all $y \in M$, $p^{-1}(y)$ has exactly n points.

(b) Prove that if N is compact, then also M is.

Exercise 4. **[Bonus exercise[1]]** Let $f : M \to N$ be a local diffeomorphism between compact manifolds. Prove that f is a covering. [Hint: For every point $x \in N$, construct an evenly covered neighborhood. Consider the inverse image $f^{-1}(x) = \{y_1, \ldots, y_n\}$, find open neigborhoods V_i of y_i where f is a diffeo, and now find a neighborhood U of x such that $f^{-1}(U) \subset \bigcup V_i$. To find U, take a neighborhood W of x such that $\overline{W} \subset \bigcap f(V_i)$, and remove something from W.]

[1] This exercise is not officially part of the exercise sheet, and it doen't influence the count of points you need to be considered over 50%. People who solve exercises 1,2,3 correclty will already get the usual 60 points given by an exercise sheet. People who try to solve exercise 4, may get some extra points added to their total