Review of perfectoid fields and almost mathematics

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May 3, 2015

We follow closely [1, Chapter 2] and [2, Chapters 3,4].

Part I
Perfectoid fields

Let p be a prime.

1 Introduction

Definition. *Let $(K, |\cdot|)$ be a valued field, complete with respect to a non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. It is a perfectoid field if*

- *$|K^\times| \subset \mathbb{R}_{\geq 0}$ is non-discrete,*
- *$char(K^\circ/\mathfrak{m}) = p$, where K° resp. $\mathfrak{m}$ are the valuation ring resp. the valuation ideal of K with respect to $|\cdot|$,*
- *the Frobenius $\Phi : K^\circ/(p) \to K^\circ/(p), \quad x \mapsto x^p$ is surjective.*

Remark. *Examples include*

- *in characteristic 0 : the p-adic completions of $\mathbb{Q}_p^{alg}, \mathbb{Q}_p(p^{p^{-\infty}}) := \bigcup_{n\geq 1} \mathbb{Q}_p(p^{1/p^n})$ and $\mathbb{Q}_p(\mu_{p^\infty}) := \bigcup_{n\geq 1} \mathbb{Q}_p(\mu_{p^n})$;*
- *in characteristic p : the t-adic completions of $\mathbb{F}_p((t))^{sep}$ and $\mathbb{F}_p((t))(t^{p^{-\infty}})$ for a variable t.*

Non-examples are p-adic completions of algebraic unramified extensions of $\mathbb{Q}_p$ (the value group is discrete).

Observation.

1. *If K is perfectoid of characteristic 0 and $p \in \mathfrak{m}$, then it naturally contains $\mathbb{Q}_p$.*
2. *K is perfectoid of characteristic p if and only if K is a perfect complete nonarchimedean field of characteristic p.*

2 The tilting construction for perfectoid fields

Definition. *Let A be a unital commutative ring, which is annihilated by p, $pA = 0$. Then $A \to A, a \mapsto a^p$ is a ring homomorphism. Set $R(A) := \varprojlim_{a \mapsto a^p} A$, the perfection of A.*

This yields a perfect ring (meaning that $x \mapsto x^p$ is bijective) with $pR(A) = 0$. Applying this construction to the valuation ring of a perfectoid field, we obtain the following interesting result.

Proposition. *Let K be a perfectoid field and $\pi \in K^\circ \setminus \{0\}$ such that $|p| \leq |\pi| < 1$. For $y \in K^\circ/(\pi)$, denote by $\hat{y}$ a lift to K°.*

1. *The maps*

$$\begin{aligned} R(K^\circ/(\pi)) &\rightleftarrows \varprojlim_{x \mapsto x^p} K^\circ \\ (x_n)_{n\geq 0} &\mapsto (lim_{m\to\infty}\ \hat{x}_{n+m}^{p^m})_{n\geq 0} \\ (y^{(n)} + (\pi))_{n\geq 0} &\leftarrow\!\shortmid (y^{(n)})_{n\geq 0} \end{aligned}$$

   are mutually inverse and establish a well-defined and natural bijection between multiplicative monoids respecting the zero elements on both sides. In particular, the limits $lim_{m\to\infty}\ \hat{x}_{n+m}^{p^m}$ exist and are independent of the choices of the lifts.

2. *The ring $R(K^\circ/(\pi))$ is an integral domain, independent of the choice of π.*

3. *Denote by $-^\sharp$ the map*

$$K^\flat := Q(R(K^\circ/(\pi))) \xrightarrow{\cong} \varprojlim_{x \mapsto x^p} K \xrightarrow{\theta_0} K$$

   where $\theta_0((y^{(n)})_n) := y^{(0)}$. Then

$$|x|_{K^\flat} := \left|x^\sharp\right|_K$$

   defines a non-archimedean absolute value

   - *which makes $K^\flat$ into a perfectoid field of characteristic p and*
   - *with respect to which $K^{\flat\circ} = R(K^\circ/(\pi))$.*

4. *If char $(K) = p$, then $K^\flat \cong K$.*

Definition. *$K^\flat$ is called the tilt of K.*

From the previous proposition, we see that, in particular, the tilt of a perfectoid field is again a perfectoid field (note that, for example, $\mathbb{Q}_p^\flat = \mathbb{F}_p$). The properties of the tilt of a perfectoid field are summarized in the following theorem:

Theorem. *Let K be a perfectoid field with tilt $K^\flat$.*

1. *The map*

$$\begin{array}{ccc} \text{set of equivalence classes} & & \text{set of equivalence classes} \\ \text{of continuous absolute values on } K & \longrightarrow & \text{of continuous absolute values on } K^\flat \\ \\ \text{equivalence class of } |\cdot| & \longmapsto & \text{equivalence class of } \left|\cdot^\sharp\right| =: |\cdot|^\flat \end{array}$$

   is bijective.

2. *Any finite extension of K is a perfectoid field.*

3. *The association $-^\flat$ induces an equivalence of categories*

$$\begin{array}{c} \text{finite extensions of } K \xrightarrow{\cong} \text{finite extensions of } K^\flat \\ L \longmapsto L^\flat \end{array}$$

   with $[L:K] = \left[L^\flat : K^\flat\right]$. In particular, K is algebraically closed if $K^\flat$ is algebraically closed.

Part II
Almost mathematics

3 Almost category theory

Fix a perfectoid field $(K, |\cdot|, K^\circ, \mathfrak{m})$, let $\mathrm{char}(K^\circ/\mathfrak{m}) = p$. Note that $\mathfrak{m}$ is a flat K°-module (since torsionfree modules over valuation rings are flat) and that $\mathfrak{m}^2 = \mathfrak{m} \cong \mathfrak{m} \otimes_{K^\circ} \mathfrak{m}$.

Definition. *Let M be a K°-module.*

- *An element $x \in M$ is almost zero if $\mathfrak{m}x = 0$. M is almost zero if $\mathfrak{m}M = 0$.*
- *A morphism $f \in Hom_{K^\circ}(M, N)$ is an almost isomorphism if $ker(f)$ and $coker(f)$ are almost zero.*

Example. *$K^\circ/\mathfrak{m}$ is almost zero, whereas $K^\circ/(p)$ is not.*

Denote by $\mathrm{Ann}(\mathfrak{m})$ the full subcategory of almost zero objects in K°-mod.

Lemma. *The category $Ann(\mathfrak{m})$ is thick (i.e. closed under subobjects, quotients and extensions).*

Proof. Let $M_1 \subset M$ and M/M_1 be almost zero. Then $\mathfrak{m}M \subseteq M_1$, hence $0 = \mathfrak{m}^2 M = \mathfrak{m}M$. □

Therefore we can form the quotient category K°-mod / $\mathrm{Ann}(\mathfrak{m}) =: K^{\circ a}$-mod. Denote the exact canonical functor K°-mod $\to K^{\circ a}$-mod by $M \mapsto M^a$. The latter object is M, seen as an object of $K^{\circ a}$-mod. We record the following facts about this abelian category:

- Let $f : M \to N$ be a morphism in K°-mod. f^a is an isomorphism if and only if f is an almost isomorphism. Hence, M lies in $\mathrm{Ann}(\mathfrak{m}) \Leftrightarrow u : M \to 0$ is an almost isomorphism $\Leftrightarrow u^a : M^a \to 0$ is an isomorphism.
- For any two K°-modules M, N, there is an equality of Hom-sets
$$\mathrm{Hom}_{K^{\circ a}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N)$$
by which the left hand side obtains the natural structure of a K°-module.
- There is no non-zero $\mathfrak{m}$-torsion in $\mathrm{Hom}_{K^{\circ a}}(X, Y)$ for any two objects X, Y in $K^{\circ a}$-mod (indeed, writing $X = M^a, Y = N^a$, then for $f \in \mathrm{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N)$ with $\mathfrak{m}f = 0$, one has $0 = \mathfrak{m}f(\mathfrak{m} \otimes_{K^\circ} M) = f(\mathfrak{m}^2 \otimes_{K^\circ} M) = f(\mathfrak{m} \otimes_{K^\circ} M)$).

Set $\mathrm{alHom}_{K^{\circ a}}(X,Y) := \mathrm{Hom}_{K^{\circ a}}(X,Y)^a$.

Proposition.

- *The tensor product in K°-mod induces a bifunctor $-\otimes_{K^{\circ a}}-$ on $K^{\circ a}$-mod, making it an abelian tensor category. There is a functorial isomorphism*

$$Hom_{K^{\circ a}}(X \otimes_{K^{\circ a}} Y, Z) \cong Hom_{K^{\circ a}}(X, alHom_{K^{\circ a}}(Y,Z)).$$

- *The functor $-^a$ has a right adjoint $-_*$ and a left adjoint $-_!$ with*

$$(X_*)^a \cong X \cong (X_!)^a$$

and

$$(M^a)_* \cong Hom_{K^\circ}(\mathfrak{m}, M) \quad \text{resp.} \quad (M^a)_! \cong \mathfrak{m} \otimes_{K^\circ} M$$

for all objects X in $K^{\circ a}$-mod and all objects M in K°-mod. Moreover, $-_*$ is left exact and $-_!$ is exact. Both are fully faithful.

In particular, the functors are defined as $X_* := \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, X)$ and $X_! := \mathfrak{m} \otimes_{K^\circ} X_*$. We have, for example, $(K^{\circ a})_! \cong \mathfrak{m}$.

4 Almost commutative algebra

Definition. *A $K^{\circ a}$-algebra A is a commutative unitary monoid object in $K^{\circ a}$-mod (i.e. there are morphisms $\mu : A \otimes_{K^{\circ a}} A \to A$ and $\eta_A : K^{\circ a} \to A$ satisfying certain associativity and commutativity constraints). Denote the category of $K^{\circ a}$-algebras by $K^{\circ a}$-alg.*

We collect several properties about almost algebras:

- $-^a$ restricts to an essentially surjective functor K°-alg $\to$ $K^{\circ a}$-alg.
- Let A be an object of $K^{\circ a}$-alg. Then there is the notion of an A-module, the category of which is denoted by A-mod. It is again an abelian tensor category.
- An A-algebra B is a $K^{\circ a}$-algebra B together with a morphism of $K^{\circ a}$-algebras $A \to B$.

Definition. *Let $R^a = A$ be a $K^{\circ a}$-algebra and X be an A-module. Then X is flat resp. almost projective if $X \otimes_A -$ resp. $alHom_A(X,-)$ is an exact functor. The module $X = M^a$ is almost finitely presented if for all $\epsilon \in \mathfrak{m}$, there exists a finitely presented R-module M_ϵ and a homomorphism $f_\epsilon : M_\epsilon \to M$, such that $\epsilon\ ker(f_\epsilon) = 0 = \epsilon\ coker(f_\epsilon)$.*
Let B be an A-algebra. The morphism $A \to B$ is étale if the following two properties are satisfied:

- *there exists* $e \in (B \otimes_A B)_*$ *such that* e *is idempotent,* $\mu \circ e = \eta_B$ *and* $ker(\mu)_* \cdot e = 0$ *(note: this is the definition of an unramified morphism);*
- B *is flat as an* A*-module.*

The morphism is finite étale if it is étale and B *is almost finitely presented as an* A*-module.*

Denote by $A_{\text{fét}}$ the category of finite étale A-algebras.
An intermediate step towards proving the main theorem about tilting, namely the equivalence of the categories $K_{\text{fét}}$ and $K^{\flat}_{\text{fét}}$, is the following.

Theorem. *Let* A *be a* $K^{\circ a}$*-algebra, which is flat over* $K^{\circ a}$ *and which is isomorphic with* $\varprojlim_n A/(\pi)^n$*. Then* $B \mapsto B \otimes_A A/(\pi)$ *induces an equivalence of categories* $A_{fét} \cong (A/(\pi))_{fét}$*.*

References

[1] Ofer Gabber, Lorenzo Ramero. *Almost Ring Theory*. Lecture Notes in Mathematics 1800, Springer, 2003.

[2] Peter Scholze. *Perfectoid Spaces*. Publ. math. de l'IHÉS 116 (2012), no. 1, 245-313.