CIRCULAR SETS OF PRIMES OF IMAGINARY QUADRATIC NUMBER FIELDS

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Abstract. Let p be an odd prime number and let K be an imaginary quadratic number field whose class number is not divisible by p. For a set S of primes of K whose norm is congruent to 1 modulo p, we introduce the notion of strict circularity. We show that if S is strictly circular, then the group $G(K_S(p)/K)$ is of cohomological dimension 2 and give some explicit examples.

1. Introduction

Let K be a number field, p a prime number and S a finite set of primes of K not containing any primes dividing p. Only little has been known on the structure of the Galois group $G(K_S(p)/K)$ of the maximal p-extension of K unramified outside S, in particular there has been no result on the cohomological dimension of $G(K_S(p)/K)$. Recently, Labute [La] showed that pro-p-groups whose presentation in terms of generators and relations is of a certain type, so-called mild pro-p-groups, are of cohomological dimension 2. If $K = \mathbb{Q}$, Labute used results of Koch on the relation structure of $G(\mathbb{Q}_S(p)/\mathbb{Q})$ and ended up with a criterion on the set S for the group $G(\mathbb{Q}_S(p)/\mathbb{Q})$ to be of cohomological dimension 2. Schmidt [S] extended the result of Labute by arithmetic methods and weakened Labute's condition on S.

The objective of this paper is to study the case where K is an imaginary quadratic number field whose class number is not divisible by p. In the first section we introduce the notions of the linking number of two primes and of strict circularity of a set of primes of K, all of this in complete analogy with the case $K = \mathbb{Q}$. Using Labute's results we obtain the criterion that if S is strictly circular then $G(K_S(p)/K)$ is a mild pro-p-group and hence of cohomological dimension 2. In the following section we give some explicit examples of strictly circular sets of primes, and in section 4 we study how a strictly circular set T can be enlarged to set S of primes of K, such that $G(K_S(p)/K)$ has cohomological dimension 2 as well.

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2. Linking numbers and strictly circular sets

Let p be an odd prime number and K an imaginary quadratic number field whose class number is not divisible by p, and which is different from $\mathbb{Q}(\sqrt{-3})$ if $p = 3$. Let $S = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ be a set of primes of K whose

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norm is congruent to 1 mod p. For a subset T of S, we denote the maximal p-extension of K unramified outside T by $K_T(p)$, and we put $G_T(p) = G(K_T(p)/K)$.

Let I_K denote the idèle group of K, and for a subset T of S let U_T be the subgroup of I_K consisting of those idèles whose components for $\mathfrak{q} \in T$ are 1 and for $\mathfrak{q} \notin T$ are units. For $\mathfrak{q} \in S$ we denote by $K_\mathfrak{q}$ the completion of K at $\mathfrak{q}$ and by $U_\mathfrak{q}$ the unit group of $K_\mathfrak{q}$. Furthermore, let $\pi_\mathfrak{q}$ be a uniformizer of $K_\mathfrak{q}$ and let $\alpha_\mathfrak{q}$ be a generator of the cyclic group $U_\mathfrak{q}/U_\mathfrak{q}^p$. Let $\mathfrak{Q}$ be an extension of $\mathfrak{q}$ to $K_S(p)$. We let $\sigma_\mathfrak{q}$ be an element of $G_S(p)$ with the following properties:

(1) $\sigma_\mathfrak{q}$ is a lift of the Frobenius automorphism of $\mathfrak{Q}$;
(2) the restriction of $\sigma_\mathfrak{q}$ to the maximal abelian subextension $\tilde{K}/K$ of $K_S(p)/K$ is equal to $(\hat{\pi}_\mathfrak{q}, \tilde{K}/K)$, where $\hat{\pi}_\mathfrak{q}$ denotes the idèle whose $\mathfrak{q}$-component equals $\pi_\mathfrak{q}$ and all other components are 1.

Let $\tau_\mathfrak{q}$ denote an element of $G_S(p)$ such that

(1) $\tau_\mathfrak{q}$ is an element of the inertia group $T_\mathfrak{Q}$ of $\mathfrak{Q}$ in $K_S(p)/K$;
(2) the restriction of $\tau_\mathfrak{q}$ to $\tilde{K}/K$ equals $(\hat{\alpha}_\mathfrak{q}, \tilde{K}/K)$, where $\hat{\alpha}_\mathfrak{q}$ denotes the idèle whose $\mathfrak{q}$-component equals $\alpha_\mathfrak{q}$ and all other components are equal to 1.

For any subset T of S, class field theory provides an isomorphism

$$I_K/(U_T I_K^p K^\times) \cong G_T(p)/G_T(p)^p[G_T(p), G_T(p)] = H_1(G_T(p), \mathbb{Z}/p\mathbb{Z}).$$

Let V_T denote the Kummer group

$$V_T = \{a \in K^\times \mid a \in K_\mathfrak{q}^{\times m} \text{ for } \mathfrak{q} \in T \text{ and } a \in U_\mathfrak{q} K_\mathfrak{q}^{\times m} \text{ for } \mathfrak{q} \notin T\}$$

We remark that due to [NSW], 8.7.2, we have an exact sequence

$$0 \to \mathcal{O}_K^\times/p \to V_\varnothing(K) \to {}_p\mathrm{Cl}(K) \to 0.$$

By our assumptions, this yields that $V_\varnothing(K) = 0$, and since $V_T(K) \subset V_\varnothing(K)$ we have $V_T(K) = 0$. This implies that the dual of the Kummer group $Б_T(K) = V_T(K)^\vee$ is trivial. The group on the left hand side of the above isomorphism is therefore given by

$$I_K/(U_T I_K^p K^\times) \cong U_\varnothing/U_T U_\varnothing^p = \prod_{\mathfrak{q} \in T} U_\mathfrak{q}/U_\mathfrak{q}^p = (\mathbb{Z}/p\mathbb{Z})^{\# T}$$

(see [Ko], §11.3). In particular, the automorphism $\tau_\mathfrak{q}$ restricts to a generator of the cyclic group $H_1(G_{\{\mathfrak{q}\}}(p), \mathbb{Z}/p\mathbb{Z})$. We use this fact for the definition of the linking numbers.

Definition 2.1. *For two primes $\mathfrak{q}_i$, $\mathfrak{q}_j \in S$, the* linking number $\ell_{ij} \in \mathbb{Z}/p\mathbb{Z}$ *of $\mathfrak{q}_i$ and $\mathfrak{q}_j$ is defined by the formula*

$$\sigma_{\mathfrak{q}_i} \equiv \tau_{\mathfrak{q}_j}^{\ell_{ij}} \mod G_{\{\mathfrak{q}_j\}}(p)^p$$

where, by abuse of notation, $\sigma_{\mathfrak{q}_i}$ and $\tau_{\mathfrak{q}_j}$, respectively, denote the images of $\sigma_{\mathfrak{q}_i} \in G_S(p)$ and $\tau_{\mathfrak{q}_j} \in G_S(p)$, respectively, in $G_{\{\mathfrak{q}_j\}}(p)$.

In other words, ℓ_{ij} is the image of the Frobenius automorphism $\sigma_{\mathfrak{q}_i} \in G_S(p)$ in $H_1(G_{\{\mathfrak{q}_j\}}(p), \mathbb{Z}/p\mathbb{Z})$ which we identify with $\mathbb{Z}/p\mathbb{Z}$ by means of its generator $\tau_{\mathfrak{q}_j}$. Note that $\ell_{ii} = 0$ for all $i = 1, \dots, n$. The linking number ℓ_{ij} is independent of the choice of the uniformizer $\pi_{\mathfrak{q}_i}$ of $K_{\mathfrak{q}_i}$ (this follows from the above isomorphism for the case $T = \{\mathfrak{q}_j\}$), but it depends on the choice of $\alpha_{\mathfrak{q}_j}$. If $\alpha_{\mathfrak{q}_j}$ would be replaced by $\alpha_{\mathfrak{q}_j}^s$, where s is prime to p, then ℓ_{ij} would be multiplied by s. The defining equation of the linking number ℓ_{ij} is equivalent to

$$\hat{\pi}_{\mathfrak{q}_i} \equiv \hat{\alpha}_{\mathfrak{q}_j}^{\ell_{ij}} \mod U_{\{\mathfrak{q}_j\}} I_K^p K^\times$$

which makes it possible to calculate the linking numbers in some examples, see section 3.

Let us pause here for a moment to explain the analogy to link theory. Assume we are given two disjoint knots I and J in S^3. Then the linking number $\mathrm{lk}(I,J)$ is defined as follows. The knot I is a loop in $S^3 - J$, hence it represents an element of $\pi_1(S^3 - J)$. After a choice of a generator of the infinite cyclic group $H_1(S^3 - J)$, $\mathrm{lk}(I,J)$ is defined as the image of I under the map

$$\pi_1(S^3 - J) \twoheadrightarrow \pi_1^{ab}(S^3 - J) \cong H_1(S^3 - J) \cong \mathbb{Z}.$$

In the number theoretical context described above, the linking number ℓ_{ij} is given by the image of the Frobenius automorphism σ_i under the map

$$\begin{aligned}\pi_1^{et}(X - S) \twoheadrightarrow \pi_1^{et}(X - \{\mathfrak{q}_j\}) \twoheadrightarrow H_1(X - \{\mathfrak{q}_j\}, \mathbb{Z}/p\mathbb{Z}) &= H_1(G_{\{\mathfrak{q}_j\}}(p), \mathbb{Z}/p\mathbb{Z}) \\ &\cong \mathbb{Z}/p\mathbb{Z}\end{aligned}$$

where $X = \mathrm{Spec}(\mathcal{O}_K)$ and we have chosen a generator of the cyclic group $H_1(X - \{q_j\}, \mathbb{Z}/p\mathbb{Z})$.

We denote by $\Gamma_S(p)$ the directed graph with vertices the primes of S and a directed edge $\mathfrak{q}_i\mathfrak{q}_j$ from $\mathfrak{q}_i$ to $\mathfrak{q}_j$ if $\ell_{ij} \neq 0$. The graph $\Gamma_S(p)$, together with the ℓ_{ij} is called the *linking diagram* of S.

Definition 2.2. *A finite set of primes of K whose norm is congruent to* 1 *modulo p is called* strictly circular with respect to p *(and $\Gamma_S(p)$ a* non-singular circuit*) if there exists an ordering $S = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ of the primes in S such that the following conditions are fulfilled:*

(1) *The vertices $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ of $\Gamma_S(p)$ form a circuit $\mathfrak{q}_1\mathfrak{q}_2 \dots \mathfrak{q}_n\mathfrak{q}_1$.*
(2) *If i, j are both odd, then $\mathfrak{q}_i\mathfrak{q}_j$ is not an edge of $\Gamma_S(p)$.*
(3) *$\ell_{12}\ell_{23} \dots \ell_{n-1,n}\ell_{n1} \neq \ell_{1n}\ell_{21} \dots \ell_{n,n-1}$.*

We remark that condition (1) implies that n is even and ≥ 4. Note that condition (3) does not depend on the choice of the $\alpha_{\mathfrak{q}_j}$. It is satisfied if there exists an edge $\mathfrak{q}_i\mathfrak{q}_j$ of the circuit $\mathfrak{q}_1\mathfrak{q}_2 \dots \mathfrak{q}_n\mathfrak{q}_1$ such that $\mathfrak{q}_j\mathfrak{q}_i$ is not an edge of $\Gamma_S(p)$.

We will now show that G has representation of *Koch type.*

Proposition 2.3 (Koch)**.** *The group $G_S(p)$ has a presentation of Koch type, i.e. we have a minimal presentation $G_S(p) = F/R$ where F is the free pro-p-group on generators $x_1, \dots, x_n$, and R is minimally generated as a normal subgroup of F by relations $r_1, \dots, r_n$ which are given modulo $F_{(3)}$ by*

$$r_i \equiv x_i^{\mathrm{N}(\mathfrak{q}_i)-1} \prod_{\substack{j=1 \\ j\neq i}}^{n} [x_i, x_j]^{\ell_{ij}} \mod F_{(3)},\ i = 1, \dots, n.$$

Here $F_{(3)}$ denotes the third step of the descending p-central series of F.

Proof. We have already seen above that $G_S(p)$ has a minimal generating system consisting of the n elements $\tau_{\mathfrak{q}_1}, \ldots, \tau_{\mathfrak{q}_n}$. The abelianization $G_S(p)^{ab}$ of $G_S(p)$ is a finitely generated abelian pro-p-group. If $G_S(p)^{ab}$ were infinite, it would have a quotient isomorphic to $\mathbb{Z}_p$, which corresponds to a $\mathbb{Z}_p$-extension K_∞ of K inside $K_S(p)$. By [NSW], Thm. 10.3.20(ii), a $\mathbb{Z}_p$-extension of K is ramified at at least one prime dividing p. This contradicts $K_\infty \subset K_S(p)$, hence $G_S(p)^{ab}$ is finite. In particular, $G_S(p)$ has at least as many relations as generators. From [NSW], 8.7.11 we obtain the inequality

$$\dim_{\mathbb{Z}/p\mathbb{Z}} H^1(G_S(p), \mathbb{Z}/p\mathbb{Z}) \geq \dim_{\mathbb{Z}/p\mathbb{Z}} H^2(G_S(p), \mathbb{Z}/p\mathbb{Z}),$$

which implies that a minimal system of generators of R as a normal subgroup of F consists of n elements. Such a system is given by the set of relations

$$r_i = x_i^{\mathrm{N}(\mathfrak{q}_i)-1}[x_i^{-1}, y_i^{-1}], \quad i = 1, \ldots, n,$$

where $y_i \in F$ denotes a preimage of $\sigma_{\mathfrak{q}_i}$, see [Ko], §11.4. The definition of the +linking numbers yields

$$y_i \equiv \prod_{\substack{j=1\\ j\neq i}}^{n} x_j^{\ell_{ij}} \mod F_{(2)}.$$

Hence we obtain

$$r_i \equiv x_i^{\mathrm{N}(\mathfrak{q}_i)-1}[x_i, y_i] \equiv x_i^{\mathrm{N}(\mathfrak{q}_i)-1}[x_i, \prod_{\substack{j=1\\ j\neq i}}^{n} x_j^{\ell_{ij}}] \equiv x_i^{\mathrm{N}(\mathfrak{q}_i)-1} \prod_{\substack{j=1\\ j\neq i}}^{n} [x_i, x_j]^{\ell_{ij}} \mod F_{(3)},$$

which finishes the proof. □

Since $G_S(p)$ is of Koch a type, a result of Labute, ([La], Thm. 1.6.), applies, which states that $G_S(p)$ is a mild pro-p-group if S is strictly circular with respect to p. Then, in particular, $G_S(p)$ has cohomological dimension 2. We summarize our considerations in the following

Theorem 2.4. *Let p be an odd prime number and let K be an imaginary quadratic number field whose class number is not divisible by p, and which is different from $\mathbb{Q}(\sqrt{-3})$ if $p = 3$. Let $S = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ be a set of primes of K whose norm is congruent to 1 mod p. Is S is strictly circular with respect to p, then $G(K_S(p)/K)$ is a mild pro-p-group and hence of cohomological dimension 2.*

3. Some examples

We use the same notation as in section 1. We let $S = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$, and denote by q_i the prime of $\mathbb{Z}$ lying below $\mathfrak{q}_i$.

We firstly consider the case where each q_i is inert in $K/\mathbb{Q}$. Then $\pi_{\mathfrak{q}_i} = q_i$ is a uniformizer of $K_{\mathfrak{q}_i}$, and an element of $U_\mathfrak{q}$ for all primes $\mathfrak{q} \neq \mathfrak{q}_i$ of K. Hence, the idèle $\hat{\pi}_{\mathfrak{q}_i}$, when considered modulo $U_S I_K^p K^\times$, is equivalent to the idèle whose $\mathfrak{q}$-component is equal to 1 for $\mathfrak{q} \notin S$ and $\mathfrak{q} = \mathfrak{q}_i$, and equal to q_i^{-1} for $\mathfrak{q} \in S \setminus \{\mathfrak{q}_i\}$. This means that, after a choice of a generator $\alpha_{\mathfrak{q}_j}$ of $U_{\mathfrak{q}_j}/U_{\mathfrak{q}_j}^p$, ℓ_{ij} is given by by

$$q_i = \alpha_{\mathfrak{q}_j}^{-\ell_{ij}} \mod U_{\mathfrak{q}_j}^p.$$

Equivalently, we can choose a primitive root ϵ_j of $\kappa_{\mathfrak{q}_j}^\times$, where $\kappa_{\mathfrak{q}_j}$ denotes the residue field of $\mathfrak{q}_j$. Then ℓ_{ij} is the image in $\mathbb{Z}/p\mathbb{Z}$ of any integer c satisfying

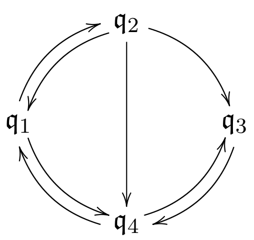
$$q_i = \epsilon_j^{-c} \mod \mathfrak{q}_j.$$

In particular, $\ell_{ij} = 0$ if and only if q_i is a p-th power modulo $\mathfrak{q}_j$. This is equivalent to q_i being a p-th power modulo q_j: if $q_i \equiv x^p \mod \mathfrak{q}_j$ for some $x \in \mathcal{O}_K$, then $q_i^2 \equiv N_{K/\mathbb{Q}}(x)^p \mod q_j$, and the claim follows. This implies in the case under consideration, that $S = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ is strictly circular with respect to p if and only if $S_\mathbb{Q} = \{q_1, \dots, q_n\}$ is strictly circular (over $\mathbb{Q}$) with respect to p.

Example 3.1. *(cf. the example after Thm 2.1 in* [S]*) Let $K = \mathbb{Q}(\sqrt{-359})$, $p = 3$. The class number of K equals 19. The prime numbers* $7, 19, 61, 163$ *are inert in $K/\mathbb{Q}$. We set*

$$\mathfrak{q}_1 = (61), \quad \mathfrak{q}_2 = (19), \quad \mathfrak{q}_3 = (163), \quad \mathfrak{q}_4 = (7)$$

and $S = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4\}$. The linking diagram has the following shape:



Hence, S is a circular set of primes and $\operatorname{cd} G(K_S(3)/K) = 2$.

In the calculations above we have made use of two things: the uniformizers $\pi_{\mathfrak{q}_i}$ have been chosen in $K^\times$, and $\pi_{\mathfrak{q}_i}$ has been a unit in $U_\mathfrak{q}$ for all $\mathfrak{q} \in S \backslash \{\mathfrak{q}_i\}$. Another case in which this is easily achieved is the case when the ideal class group of K is trivial. Then we can take a generator of $\mathfrak{q}_j$ as the uniformizer $\pi_{\mathfrak{q}_j}$ and ℓ_{ij} can be obtained from the same equations as above with q_j replaced by $\pi_{\mathfrak{q}_j}$.

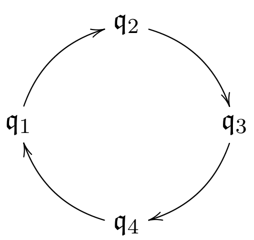
Example 3.2. *Let $K = \mathbb{Q}(i)$, $p = 3$. We put*

$$\mathfrak{q}_1 = (2 + 15i), \qquad \mathfrak{q}_2 = (4 + 15i), \quad \mathfrak{q}_3 = \overline{\mathfrak{q}}_1, \quad \mathfrak{q}_4 = \overline{\mathfrak{q}}_2$$

and $S = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4\}$. Then we have $q_1 = q_3 = 229$, $q_2 = q_4 = 241$, and we set

$$\pi_{\mathfrak{q}_1} = 2 + 15i, \quad \pi_{\mathfrak{q}_2} = 4 + 15i, \quad \pi_{\mathfrak{q}_3} = \overline{\pi}_{\mathfrak{q}_1}, \quad \pi_{\mathfrak{q}_4} = \overline{\pi}_{\mathfrak{q}_2}$$

The linking diagram has the following shape:



Hence $\operatorname{cd} G(K_S(3)/K) = 2$. *Note that, by* [Ko], *Ex. 11.15, $G(\mathbb{Q}_{\{q_1,q_2\}}(3)/\mathbb{Q})$ is finite.*

The last example raises the following question. There are no examples known of prime numbers q_1, q_2 congruent to 1 modulo p where one can show that the cohomological dimension of $G(\mathbb{Q}_{\{q_1,q_2\}}(p)/\mathbb{Q})$ equals 2. Is it possible to obtain such an example by considering strictly circular sets of primes $\{\mathfrak{q}_1, \mathfrak{q}_2, \bar{\mathfrak{q}}_1, \bar{\mathfrak{q}}_2\}$ of an imaginary quadratic number field K of class number one, in combination with some kind of descent argument? Unfortunately, the answer to this question is negative as the following considerations show. Let q_1, q_2 be prime numbers congruent to 1 modulo p, and assume there exists an imaginary quadratic number field of class number one in which q_1, q_2 are completely decomposed:

$$q_1\mathcal{O}_K = \mathfrak{q}_1\mathfrak{q}_3, \quad q_2\mathcal{O}_K = \mathfrak{q}_2\mathfrak{q}_4.$$

This definition of the primes $\mathfrak{q}_i$ implies (for an appropriate choice of the primitive roots) the following equations for the linking numbers:

$$\ell_{12} = \ell_{34},\ \ell_{23} = \ell_{41},\ \ell_{13} = \ell_{31},\ \ell_{24} = \ell_{42}.$$

Since we want to avoid that the group $G(\mathbb{Q}_{\{q_1,q_2\}}(p)/\mathbb{Q})$ is finite, we have to make sure that the conditions of [Ko], Ex. 11.15 are not fulfilled, and therefore we have in addition to assume that q_1 is a p-th power modulo q_2 and that q_2 is a p-th power modulo q_1. It is easily seen that this puts the following restraints on the linking numbers:

$$\ell_{12} + \ell_{32} = 0,\ \ell_{14} + \ell_{34} = 0,\ \ell_{21} + \ell_{41} = 0,\ \ell_{23} + \ell_{43} = 0.$$

If ρ_i denotes the initial form of the image of r_i in the graded Lie algebra associated to the descending p-central series of F, the above conditions yield the equation

$$\ell_{23}\rho_1 - \ell_{12}\rho_2 + \ell_{23}\rho_3 - \ell_{12}\rho_4 = 0.$$

This means that the sequence $\rho_1, \dots, \rho_4$ is not strongly free (cf. the definition of strong freeness in [La]), which implies, in particular, that the set $\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4\}$ is not strictly circular, and this holds true as well if we make a different choice of the primitive roots.

4. Enlarging the set of primes

Proposition 4.1. *Let p be an odd prime number and and K an imaginary quadratic number field whose class number is not divisible by p, and which is different from $\mathbb{Q}(\sqrt{-3})$ if $p = 3$. Let $S = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ be a set of primes of K whose norm is congruent to 1 mod p. If $\operatorname{cd} G(K_S(p)/K) \le 2$, then the scheme $X = \operatorname{Spec}(\mathcal{O}_K) - S$ is a $K(\pi, 1)$ for the étale topology, i.e. for any discrete p-primary $G(K_S(p)/K)$-module M, considered as a locally constant étale sheaf on X, the natural homomorphism*

$$H^i(G(K_S(p)/K), M) \to H^i_{et}(X, M)$$

is an isomorphism for all i.

Proof. We put $G = G(K_S(p)/K)$. In the same way as in the proof of [S], Prop. 3.2., the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G, H^q_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})) \Rightarrow H^{p+q}_{et}(X, \mathbb{Z}/p\mathbb{Z}),$$

where $\tilde{X}$ denotes the universal p-covering of X, implies isomorphisms

$$H^i(G, \mathbb{Z}/p\mathbb{Z}) \cong H^i_{et}(X, \mathbb{Z}/p\mathbb{Z}), \quad i = 0, 1$$

and a short exact sequence

$$0 \to H^2(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\phi} H^2_{et}(X, \mathbb{Z}/p\mathbb{Z}) \to H^2_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})^G \to 0.$$

We set $\bar{X} = \operatorname{Spec} \mathcal{O}_K$. By the flat duality theorem of Artin-Mazur, ([Mi], III, Thm. 3.1), we have

$$H^3_{et}(\bar{X}, \mathbb{Z}/p\mathbb{Z}) = \operatorname{Hom}_{\bar{X}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)^\vee = 0$$

and

$$H^2_{et}(\bar{X}, \mathbb{Z}/p\mathbb{Z})^\vee = \operatorname{Ext}^1_{\bar{X}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m),$$

the latter group sitting in an exact sequence

$$0 \to \mathcal{O}_K^\times/p \to \operatorname{Ext}^1_{\bar{X}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m) \to {}_p\operatorname{Cl}(K) \to 0.$$

Our assumptions on K implies

$$H^2_{et}(\bar{X}, \mathbb{Z}/p\mathbb{Z}) = 0.$$

The excision sequence for the pair $(\bar{X}, X)$ yields an isomorphism

$$H^2_{et}(X, \mathbb{Z}/p\mathbb{Z}) = \bigoplus_{\mathfrak{q} \in S} H^3_{\{\mathfrak{q}\}}(\operatorname{Spec} \mathcal{O}^h_{\mathfrak{q}}, \mathbb{Z}/p\mathbb{Z}),$$

where $\mathcal{O}^h_{\mathfrak{q}}$ denotes the henselization of the local ring of $\bar{X}$ at $\mathfrak{q}$. The local duality theorem ([Mi], II, Thm. 1.8) gives

$$H^3_{\{\mathfrak{q}\}}(\operatorname{Spec} \mathcal{O}^h_{\mathfrak{q}}, \mathbb{Z}/p\mathbb{Z}) \cong \operatorname{Hom}_{\operatorname{Spec} \mathcal{O}^h_{\mathfrak{q}}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)^\vee.$$

As we have assumed that for all $\mathfrak{q} \in S$, the norm of $\mathfrak{q}$ is congruent to 1 modulo p, we obtain $\dim_{\mathbb{Z}/p\mathbb{Z}} H^2_{et}(X, \mathbb{Z}/p\mathbb{Z}) = n$. Hence, by the proof of Lemma 2.3, ϕ is an isomorphism, and therefore

$$H^2_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})^G = 0.$$

The proof is then concluded as in [S], Prop. 3.2. □

Theorem 4.2. *Let p be an odd prime number and let K be an imaginary quadratic number field whose class number is not divisible by p, and which is different from $\mathbb{Q}(\sqrt{-3})$ if $p = 3$. Let S be a set of primes of K whose norm is congruent 1 mod p. Assume that $\operatorname{cd} G(K_S(p)/K) = 2$. Let $\mathfrak{l} \notin S$ be a prime whose norm is congruent to 1 modulo p, and which does not split completely in the extension $K_S(p)/K$. Then*

$$\operatorname{cd} G(K_{S \cup \{\mathfrak{l}\}}(p)/K) = 2.$$

Proof. The proof is the same as the proof of [S], Thm. 2.3, we just have to replace Prop. 3.2. of (loc.cit.) by Prop. 4.1. above. □

Corollary 4.3. *Assume that S contains a strictly circular subset T such for each $\mathfrak{q} \in S \setminus T$ there exists an edge from $\mathfrak{q}$ to a prime of T. Then* $\operatorname{cd}(G(K_S(p)/K)) = 2$.

Proof. We only need to remark that if we are given a prime $\mathfrak{q} \in S$ such that the linking number of $\mathfrak{q}$ and a certain prime $\mathfrak{l}$ of T is nontrivial, then $\mathfrak{q}$ does not split completely in $K_T(p)/K$. To see this, we fix an extension $\mathfrak{Q}$ of $\mathfrak{q}$ to $L = K_{\{\mathfrak{l}\}}(p)^{ab}$. Since the linking number of $\mathfrak{q}$ and $\mathfrak{l}$ is nontrivial, the Frobenius of $\mathfrak{Q}$ in L/K generates the whole Galois group $G(L/K) \cong \mathbb{Z}/p\mathbb{Z}$. Hence $\mathfrak{q}$ does not split completely in L/K, which proves the claim. □

Example 4.4. *Let* $K = \mathbb{Q}(\sqrt{-359})$, $p = 3$. *The prime number* $l = 113$ *is inert in* $K/\mathbb{Q}$, *and if we put* $\mathfrak{q}_5 = l\mathcal{O}_K$, *and* $S = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5\}$ *where* $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4$ *are given as in Example 3.1, the linking diagram looks as follows:*

$$\begin{tikzcd}
& \mathfrak{q}_2 \arrow[dl, bend right] \arrow[dr, bend left] \arrow[dd] & & \mathfrak{q}_5 \arrow[dl] \\
\mathfrak{q}_1 \arrow[ur, bend left] \arrow[dr, bend right] & & \mathfrak{q}_3 \arrow[dl, bend left] & \\
& \mathfrak{q}_4 \arrow[ul, bend right] \arrow[ur, bend right] & &
\end{tikzcd}$$

Hence, by Cor. 4.3 we have $\operatorname{cd} G(K_S(p)/K) = 2$ *(although* S *is not strictly circular with respect to* p*).*

Example 4.5. *Let* $K = \mathbb{Q}(\sqrt{-359})$, $p = 3$ *and* $S = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4\}$, *where* $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4$ *are given as in Example 3.1. Wet set* $\mathfrak{l} = (37, 14+\sqrt{-359})$. *Note that* $\mathfrak{l}|37$, *and* 37 *is completely decomposed in* $K/\mathbb{Q}$. *The unique subfield* L *of degree 3 over* K *of the extension* $K(\mu_7)/K$ *is a subfield of* $K_S(p)/K$, *and the prime* $\mathfrak{l}$ *of* K *is inert in* L. *Therefore, we obtain by Thm. 4.2 that* $\operatorname{cd} G(K_{S\cup\{\mathfrak{l}\}}(p)/K) = 2$.

Another result from [S] which carries over to our situation with identical proof is given by the following theorem.

Theorem 4.6. *Let* p *be an odd prime number and let* K *be an imaginary quadratic number field whose class number is not divisible by* p, *and which is different from* $\mathbb{Q}(\sqrt{-3})$ *if* $p = 3$. *Let* S *be a set of primes of* K *whose norm is congruent to* $1 \bmod p$. *Assume that* $G(K_S(p)/K) \neq 1$ *and* $\operatorname{cd} G(K_S(p)/K) \leq 2$. *Then* $\operatorname{scd} G(K_S(p)/K) = 3$ *and* $G(K_S(p)/K)$ *is a pro-*p *duality group.*

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