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Rational points on Fano varieties

the Ax-Conjecture after Kollár, and the Graber–Harris–Starr Theorem

Hauptseminar WS2013/14 — Arithmetische Geometrie

INF 288, HS 4 — Di 11:15–12:45 — 1^{st} meeting: October 15, 2013

Brief description of the seminar

The aim of this seminar will be to understand various modern algebraic geometric methods in finding rational points on varieties over various fields. Namely, we will study properties of fields over which certain types of varieties (complete intersections, hypersurfaces) are all guaranteed to have a rational point if they satisfy various degree conditions, but also on the other hand varieties whose geometric properties (e.g. Fano or rationally connected) guarantee the existence of a rational point over various fields.

More precisely there are two results in the focus of the seminar. Firstly a result of Kollár which proves that every PAC field in characteristic zero is C_1, formerly a conjecture of Ax.

Definition 1. A C_1**-field** is a field k for which every homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree $\leq n$ has a non-trivial zero.

Examples of C_1 fields are finite fields (Chevalley-Warning) and function fields of curves over algebraically closed fields (Tsen). By definition, Fano hypersurfaces over a C_1-field have a point.

Definition 2. A **PAC (pseudo-algebraically closed) field** is field k for which every geometrically integral k-variety has a k-point.

Theorem 3 ([Kol07])**.** *Every PAC field of characteristic zero is* C_1.

The proof of this theorem is geometric in nature, constructing a family of Fano varieties and cleverly using vanishing and connectedness theorems to conclude. The result is restricted to characteristic 0 mainly by an initial use of resolution of singularities, but also the vanishing theorem fails in positive characteristic in general.

In the second section we will discuss the fact that Fano varieties are rationally connected and that even this, broader, class of varieties also has a rational point over the function field of a curve. More precisely, the second result is the celebrated theorem of Graber-Harris-Starr proving that a rationally connected variety over the function field of a curve over an algebraically closed field of characteristic zero has a rational point.

Definition 4. A smooth projective variety is rationally connected if there passes a rational curve through any two points.

The fundamental examples being projective space and more generally rational varieties. The latter class does not satisfying various nice geometric properties one would want, so the definition of a rationally connected variety becomes natural and mimicks the fundamental property of projective space saying that "it contains a lot of rational curves".

Theorem 5 ([GHS03]). *Let K be the function field of a curve over $\mathbb{C}$. Then any rationally connected variety X over K has a K-point.*

After introducing standard objects in algebraic geometry such as the Hilbert, Hom and Kontsevich schemes, we will study their local properties and deformation theory and give a proof of the above theorem using an algebraic refinement of de Jong–Starr [dJS03].

Schedule – Talks

Part I: Ax's Conjecture: [Kol07]

Talk 1: C_1 and PAC fields, examples — N.N. (october 23)

Definition and examples of C_1-fields and PAC-fields: proof of Chevalley–Warning (finite fields are C_1, see [Ser73] I.2.2 Theorem 3 and Corollary 1, or [Sta08] Chapter 2 Theorem 1.3), proof of Tsen's theorem (function fields of curves over algebraically closed fields are C_1, see [Kol96] IV.6.5, or [NSW08] VI.5 Theorem 6.5.3 and Corollary 6.5.5, or [Sta08] 2.2), illustrating PAC-fields: infinite fields that are algebraic over a finite field [FJ05] Corollary 10.5, conjecturally $\mathbb{Q}^{\mathrm{solv}}$, and time permitting [FJ05] Theorem 16.18 without proof.

Talk 2: Ax's conjecture and Kollar's first reductions — N.N. (october 30)

Sections 1-5 of [Kol07]. Importance of geometric integrality: why the main theorem is not obvious (PAC guarantees points only for geometrically integral varieties, Example 4 of [Kol07]). Elaborate on definition of Fano varieties, degree vs number of variables formula for complete intersection ([Kol96] Chapter V, Definition 1.1 and Example 1.2, definition and condition when smooth copmplete intersections in $\mathbb{P}^n$ are Fano, Fano's are discussed later in talk 8 with respect to rational curves), statement of Theorems 1-3 of [Kol07], construction of family.

Talk 3: Kawamata-Viehweg vanishing — N.N. (november 5+12)

For [Kol07] Theorem 8 we need [KM98] Corollary 2.68. Firstly, prove Kodaira vanishing [KM98] Theorem 2.47: assuming decomposition theorem, that is Theorem 2.1 in [DI87], prove Corollary 2.8 of [DI87] which gives [KM98] Theorem 2.47, alternative reference: [Laz04] I.4. Secondly, set up cyclic-cover trick for the generalisation [KM98] Theorem 2.64: define (strict) normal crossing divisors (see [Kol07] Definition 7) and discuss ramified cyclic covers (from line bundles) from [KM98] 2.49–2.52, in particular Lemma 2.51 (see also [BHPVdV04] I.16-18). Thirdly, prove Theorem 2.64 and Corollary 2.68 of [KM98]: introduce $\mathbb{Q}$-Cariter divisors ([Deb01] Section 1.1), nef and big divisors ([KM98] 2.58–2.63 but only in as much as it is used for Corollary 2.68, see also [Deb01] 1.29), then prove Theorem 2.64 and Corollary 2.68 of [KM98].

Talk 4: The connectedness theorem — N.N. (november 19)

Sections 6-9 of [Kol07]. Motivate[1] connectedness as in Section 6, prove Theorem 9 of [Kol07] using Kawamata–Viehweg vanishing as proved in talk 3.

Talk 5: Proof of main result — N.N. (november 26)

Sections 10-13 of [Kol07]. Proof of main theorem, i.e., Theorem 3 of [Kol07] (for the mention of Kleiman's open ampleness result in the proof of Corollary 10 in [Kol07] we just need it for $\mathbb{Q}$-divisors, so [Laz04] Proposition 1.3.7 suffices and should be proved). Time permitting discuss what fails in positive characteristic (Remark 17 in [Kol07]).

[1]There is a typo on p. 239: better $E = B - A$, and on page 240: better $-(E - \lambda_0 F_0) = \sum_i a_i(\lambda) P_i = A - B + \Delta$.

Part II: Graber-Harris-Starr: [GHS03]

Talk 6: Hilbert, Hom and Kontsevich moduli — N.N. (december 3)

Define Hilbert scheme functor Hilb, state its existence (no proofs) [ACG11], [Kol96], [FGI+05]. Describe the Hom scheme as subscheme of the Hilbert scheme of a product. Stable curves and the Kontsevich moduli as in [GHS03] Definition 2.2 and what is used in [GHS03] after Definition 2.3, see also [FP97], in particualr irreducibility of $M_{g,0}(B, d)$, use [ACG11] XXI.11 or [Ful69], not a full proof but discussion of Hurwitz scheme (actually the case $B = \mathbb{P}^1$) and branch loci map as a covering map of the configuration space of d points on B, see also [FP97].

Talk 7: Deformations of morphisms and combs — N.N. (december 10)

Follow Chapter 2 of [Deb01], proving the formula for the tangent space in [Deb01] Proposition 2.4 stating (and time permitting proving) [Deb01] Theorem 2.6. Smoothing of combs from Proposition 4.26 in [Deb01] Section 4.6 or the corresponding section in [Kol96] II.7.

Talk 8: Rationally connected varieties — N.N. (december 17)

Characteristic zero only: definitions of various kinds of rationally connectedness: very free curve if and only if rationally connected (RC), Fanos are RC (sketch of proof, this at least needs a discussion (with pictures!) of rationally chain connected (RCC) if and only if RC). Chapter IV of [Deb01], Sections IV.1 and 3 of [Kol96], or the great notes Araujo-Kollár [AK03] and Jason Starr [Sta08]. Present the introduction of [GHS03]: main theorem and applications.

Talk 9: The de Jong-Starr family — N.N. (january 14+21)

Part 1: Section 1 of [dJS03] assuming the semi-stable reduction theorem. Part 2: Sections 2 and 3 of [dJS03] assuming but stating (in modified form) the result of [BLR95] Theorem 2.1′.

Talk 10: Proof of main theorem — N.N. (january 28)

Flexible curves, Section 2.1 of [GHS03] and the reduction to $B = \mathbb{P}^1$ from Section 3.2. Our strategy differs from [GHS03]: using talk 9 we may assume that $X \to B$ has no multiple fibres, so Bertini (beginning of [GHS03] Section 3.1) yields a quasi-section that avoids singularities of the fibres and thus is simplely ramified over B: a pre-flexible curve. Then the *first construction* (Section 2.1 of [GHS03]) shows, namely adding teeth to form a comb and then deform (Hypothesis 2.7 – Lemma 2.9; uses talk 7), that pre-flexible curves are flexible. Finally, flexible curves are enough by Proposition 2.4 based on our knowledge of the Hurwitz space $M_{g,0}(B, d)$ from talk 6 and in particular the properties of the branch locus map.

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