Seminar program tame cohomology

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Summary

For a smooth variety over the complex numbers and a prime number ℓ, étale and analytic cohomology with coefficients $\mathbb{Z}/\ell\mathbb{Z}$ coincide. If the characteristic p of the base field is positive but different from ℓ, the étale cohomology groups with coefficients $\mathbb{Z}/\ell\mathbb{Z}$ retain the same good properties as over $\mathbb{C}$. For instance, there are finiteness theorems, cohomological purity, and a smooth base change theorem. The cohomology groups are homotopy invariant and the Künneth formula holds (not only for cohomology with compact support). All this breaks down, however, if base field and coefficient ring have the same characteristic. There is overwhelming evidence that these problems are due to the existence of wild ramification "at the boundary of X". To give an example consider the first cohomology group $H^1_{\mathrm{et}}(\mathbb{A}^1_{\bar{\mathbb{F}}_p}, \mathbb{Z}/p\mathbb{Z})$, which classifies finite étale coverings of X. It is infinite dimensional because of the huge amount of étale coverings of $\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ wildly ramified in ∞.

For assertions concerning the fundamental group this problem has been adressed by introducing the tame fundamental group (see [KS10]). Under suitable regularity assumptions the tame fundamental group is topologically finitely generated and the specialization map of the tame fundamental group is at least surjective ([SGA1], VIII 2.11). Moreover, the tame fundamental group satisfies the Künneth formula ([Hos09] and there is a Lefschetz-Theorem ([EK16]).

In many cases, however, the tame fundamental group does not suffice to encode the mathematical problem in question. The following provides an example: Let X be a smooth scheme over a separably closed field k of characteristic p and $\ell \neq p$ a prime number. After [MVW06], Thm. 10.9, Suslin's singular homology with coefficients $\mathbb{Z}/\ell\mathbb{Z}$ is dual to the étale cohomology of $\mathbb{Z}/\ell\mathbb{Z}$:

$$H_i^S(X, \mathbb{Z}/\ell\mathbb{Z})^* \cong H^i_{\acute{e}t}(X, \mathbb{Z}/\ell\mathbb{Z}).$$

For $\ell = p$ the above (co)homology groups are not dual to one another, in general. For instance, $H^1_{\acute{e}t}(\mathbb{A}^1_k, \mathbb{Z}/p\mathbb{Z})$ is infinite dimensional while $H^1_S(\mathbb{A}^1_k, \mathbb{Z}/p\mathbb{Z}) = 0$. It turns out, however, (see [GS16]), that

$$H^1_S(X, \mathbb{Z}/p\mathbb{Z})^* \cong \mathrm{Hom}(\pi_1^t(X, \bar{x}), \mathbb{Z}/p\mathbb{Z}),$$

where $\pi_1^t(X, \bar{x})$ denotes the curve-tame fundamental group. The search for a comparable isomorphism in higher degrees raises the question whether there exists a tame site X_t whose fundamental group coincides with the curve-tame fundamental group such that for all i we have

$$H^i_S(X, \mathbb{Z}/p\mathbb{Z}) \cong H^i(X_t, \mathbb{Z}/p\mathbb{Z}). \tag{1}$$

One would furthermore expect that the tame cohomology groups satisfy finiteness theorems and a version of cohomological purity and smooth base change. The Künneth formula should be true without restrictions and the tame cohomology groups should be homotopy invariant.

In this seminar we study the tame site of scheme following mainly [HS21]. We will see that many of the above mentioned problems have already been adressed but there is still a lot of work to be done.

Talks

Talk 1: Introduction

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Talk about étale cohomology in characteristic $p > 0$ and explain the difficulties with p-torsion coefficients. Explain what to expect from a tame cohomology theory.

Talk 2: Valuations

Give an overview of valuations of a ring. Explain how they can be viewed as valuations of the residue field at some prime of the ring. Define the support and the center of a valuation. For a separable algebraic extension of valued fields define the inertia and the ramification group. Give some examples, e.g. classify all valuations of $\mathbb{F}[T]$ for a finite field $\mathbb{F}$ and explain the difficulties in higher dimension, e.g. for $\mathbb{F}[T, S]$.

Talk 3: Definition of the tame site

Define the algebraic tame site as introduced in [HS21] and the adic tame site (see [Hüb21], section 3). For a scheme X over a base scheme S define the adic space $\mathrm{Spa}(X, S)$ (see [Hüb21], Lemma 2.1 and) and its tame site ([Hüb21] end of section 3). Explain excision and continuity. This covers sections 2,3, and 4 of [HS21]. See also [Wed19] for an introduction to adic spaces, we only need the case where the rings are endowed with the discrete topology.

Talk 4: The tame fundamental group

Define the various notions of tameness listed in [KS10], section 4. State the key lemma ([KS10], 2.4) and explain the idea of its proof. Show the equivalent of the different notions of tameness as in [KS10], Theorem 4.4 Finally prove that the curve tame fundamental group is isomorphic to the fundamental group of the tame site ([HS21], Proposition 5.2). Note that there is a mistake in the proposition. In mixed characteristic one has to assume resolution of singularities.

Talk 5: Comparison with Čech cohomology, algebraic version

Present sections 6 and 7 of [HS21]. The material is based on Artin's article about joins of Hensel rings which investigates when étale cohomology can be computed by Čech cohomology. In [HS21] the Artin's results are used to prove a tame version.

Talk 6: Comparison with étale cohomology and finiteness in dimension 1

This is a short talk covering sections 8 and 9 of [HS21]. Following section 8 show that tame cohomology coincides with étale cohomology in the expected cases. Moreover prove that the cohomology groups are finite for schemes of dimension 1 (section 9).

Talk 7: Joins of Henselian Huber pairs

Discuss specializations and henselian Huber pairs following [HS21], section 10. Present section 11 of [HS21], which is a generalization to Huber pairs of Artin's result about joins of Hensel rings.

Talk 8: Comparison with Čech cohomology, adic version

Introduce Riemann Zariski morphisms of adic spaces and discuss their properties (section 12 of [HS21]). Then explain how tame cohomology of a discretely ringed adic space can be computed by Čech cohomology (section 13 of [HS21]). Assemble all the parts of previous talks to prove the comparison of the tame cohomology of the adic and algebraic tame sites (section 14 of [HS21]). Finally explain the purity and homotopy invariance results from [HS21], section 14.

Talk 9: Suslin homology

This talk gives a glimpse of a future application of tame cohomology. Motivate why we expect a connection of tame cohomology and Suslin homology. Present the construction of the comparison homomorphism (section 15 of [HS21]).

References

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