

# PROPERTIES OF STURM'S FORMULA

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ABSTRACT. In contrary to large weights, Sturm's operator fails to represent the holomorphic projection operator for small weights in arbitrary Siegel genus. We emphasize the roles of continuation of Poincaré series, holomorphic projection, Sturm's formula, and spectral analysis in this context, and explain their interaction.

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This article summarizes a series of papers [5]–[8] on the program of realizing holomorphic projection in the Siegel case of arbitrary genus and small weight. Parts of these results are joint with R. Weissauer ([7],[8]). We also contain a sketch of ongoing projects as well as some expectations we have on this subject. Each of the papers [5]–[8] has special aims on its own. But their common intention is to understand Sturm's operator and its connection with the holomorphic projection operator from a conceptional representation theoretic point of view. For this reason we enlarge the genus of the special linear group  $\mathrm{SL}_2$  and study symplectic groups instead, and we keep the weight small.

## 1. HOLOMORPHIC PROJECTION

In order to use the properties of holomorphic automorphic forms it is often necessary to reduce  $C^\infty$ -forms to their holomorphic part. To have a precise notion of this holomorphic projection, we describe it as  $L^2$ -operator. Let  $G_m = \mathrm{Sp}_m(\mathbb{R})$  be the real symplectic group of genus  $m$ . Let  $\Gamma$  be a subgroup of finite index in  $\mathrm{Sp}_m(\mathbb{Z})$  which contains the subgroup of translations  $\Gamma_\infty$ . This last condition is made to keep Fourier expansions as simple as possible. It can be removed by standard technics. Then  $L^2(\Gamma \backslash G_m)$  is a Hilbert space with unitary

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2010 *Mathematics Subject Classification.* 11F46, 11F70, 46G20.

This work was supported by the European Social Fund.

action of  $G_m$  by right translations. Let  $\rho$  be an irreducible representation of the maximal compact group  $K_m$  of  $G_m$ . Let  $L^2(\Gamma \backslash G_m)_\rho$  be the subspace of functions of  $K_m$ -type  $\rho$ . Then the orthogonal projection operator

$$S : L^2(\Gamma \backslash G_m)_\rho \xrightarrow{\text{proj}} L^2(\Gamma \backslash G_m)_{\rho, \text{hol}}$$

describes the holomorphic projection. Here  $L^2(\Gamma \backslash G_m)_{\rho, \text{hol}}$  is the holomorphic part of the spectrum, where a  $C^\infty$ -function is called holomorphic if under the action of the Lie algebra  $\mathfrak{g}_m^\mathbb{C}$  it is annihilated by the minus part  $\mathfrak{p}^-$  in the Cartan decomposition  $\mathfrak{g}_m^\mathbb{C} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ . The latter depends on a system of positive roots which is chosen to be compatible with the following.

It is often convenient to work with functions on the Siegel upper half space  $\mathcal{H}_m$  rather than on  $G_m$ . Let  $J : G_m \times \mathcal{H}_m \rightarrow \text{GL}_m(\mathbb{C})$  be given by  $J\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) = CZ + D$ . Restricted to  $K_m$ , we get an isomorphism  $J(\cdot, i) : K_m \xrightarrow{\sim} U(m)$  we rely on for the whole paper. Let  $C^\infty(\mathcal{H}_m, V_\rho)$  be the space of  $C^\infty$ -functions on  $\mathcal{H}_m$  with values in  $V_\rho$ , and let  $C^\infty(G_m, V_\rho) = C^\infty(G_m) \otimes V_\rho$ . There is an isomorphism

$$\begin{aligned} C^\infty(\mathcal{H}_m, V_\rho) &\xrightarrow{\sim} C^\infty(G, V_\rho)_\tau, \\ f(Z) &\mapsto F(G) = \rho^{-1}(J(g, i))f(gK \cdot i). \end{aligned}$$

Here  $\tau = \rho^{-1} \circ J(\cdot, i)$ , and we identify the  $U(m)$ -representation  $(\rho, V_\rho)$  with the corresponding representation of  $\text{GL}_m(\mathbb{C})$ . Under this isomorphism, the action of  $\mathfrak{p}^-$  corresponds to the anti-holomorphic differential operator  $\partial_{\bar{Z}}$ . Normalizing the measures  $dk$  and  $dg$  of  $K$  and  $G_m$  such that  $dg = dV_{\text{inv}}dk$ , where  $dV_{\text{inv}}$  is the invariant measure on  $\mathcal{H}_m$ , we have the following identity of scalar products. For  $C^\infty$ -cusp forms  $f$  and  $h$  of weight  $\rho$  for  $\Gamma$  it holds

$$\langle f, h \rangle = \langle \langle F, H \rangle \rangle_{L^2(\Gamma \backslash G_m)}.$$

For simplicity we assume  $\rho = \det^\kappa$  to be scalar for the rest of this paragraph. As the holomorphic projection  $S$  is orthogonal, it is characterized by

$$\langle f, h \rangle = \langle S(f), h \rangle$$

for all holomorphic cusp forms  $h \in [\Gamma, \kappa]_0$  of weight  $\kappa$ . Now assume we have a system of Poincaré series  $p_\tau$  in  $[\Gamma, \kappa]_0$ , where  $\tau$  is half-integral and positive definite, so that the Fourier coefficients  $a(\tau)$  of  $S(f) \in [\Gamma, \kappa]_0$  are given by

$$\langle S(f), p_\tau \rangle = a(\tau)c(m, \kappa).$$

Here  $c(m, \kappa)$  is a constant independent of  $\tau$ . As this must equal  $\langle f, p_\tau \rangle$  for the non-holomorphic form  $f$ , we look at its Fourier expansion

$$f(Z) = \sum_{\tau} a(\tau, Y) e^{2\pi i \text{tr}(\tau X)}$$

with  $Z = X + iY$ . By unfolding we find

$$\langle f, p_\tau \rangle = \det(\tau)^{-\frac{m+1}{2}} \int_{Y>0} a(\tau, Y) e^{-2\pi \text{tr}(Y)} \det(Y)^\kappa \frac{dY_{\text{inv}}}{\det(Y)^{\frac{m+1}{2}}},$$

where  $dY_{inv} = \det(Y)^{-\frac{m+1}{2}} \prod_{j \leq k} dy_{jk}$  for  $Y = (y_{jk})_{jk}$ . So turning things around, defining

$$(1) \quad a(\tau) := c(m, \kappa)^{-1} \det(\tau)^{-\frac{m+1}{2}} \int_{Y>0} a(\tau, Y) e^{-2\pi \operatorname{tr}(Y)} \det(Y)^\kappa \frac{dY_{inv}}{\det(Y)^{\frac{m+1}{2}}},$$

we obtain the Fourier expansion  $\sum_{\tau>0} a(\tau) e^{2\pi i \operatorname{tr}(\tau Z)}$  of the holomorphic projection  $S(f)$ . The resulting operator

$$St_\kappa : \sum_{\tau} a(\tau, Y) e^{2\pi i \operatorname{tr}(\tau X)} \mapsto \sum_{\tau>0} a(\tau) e^{2\pi i \operatorname{tr}(\tau Z)},$$

where  $a(\tau)$  is defined by (1), is called **Sturm's operator**. It is not only defined on  $L^2$ -functions, but also for all non-holomorphic modular forms of moderate growth, i.e. forms such that the integral (1) exists.

**Theorem 1.1.** *In the following cases Sturm's operator realizes the holomorphic projection operator  $St_\kappa = S$ .*

- (i) [12], [13] *In the classical case of genus  $m = 1$  and large weight  $\kappa > 2$ .*
- (ii) [10] *For arbitrary genus  $m$  and large weight  $\kappa > 2m$ .*
- (iii) [3] *For genus  $m = 1$  and weight  $\kappa = 2m = 2$ .*
- (iv) [5] *For genus  $m = 2$  and weight  $\kappa = 2m = 4$ .*

To illustrate the significance of holomorphic projection we shortly recall that first, in [3] it was an important tool to interpret certain convolution  $L$ -series. Second, it was used in [14] to reveal the true nature of mock modular forms as the holomorphic parts of weak harmonic Maass forms.

## 2. POINCARÉ SERIES

The whole argument in Section 1 relies on a given system of Poincaré series in the space  $[\Gamma, \kappa]_0$  of holomorphic cusp forms for  $\Gamma$  of weight  $\kappa$ . That is a system of functions generating  $[\Gamma, \kappa]_0$  and producing the Fourier coefficients via the inner product. The theory of Poincaré series was studied systematically in [11]. For large weights  $\kappa > 2m$  Panshichkin [10] introduced such systems tracking back to definitions by Neuenhöffer [9]. For matters of applications we prefer to work with Poincaré series of exponential type also tracking back to [9] and first introduced by Klingen [4],

$$P_\tau(g) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \det(J(\gamma g, i))^{-\kappa} e^{2\pi i \operatorname{tr}(\tau(\gamma g \cdot i))}.$$

These (as all) Poincaré series converge if and only if  $\kappa > 2m$ . From a purely analytic point of view this is the reason why things become interesting for small weights. For those the common procedure is to analytically continue these Poincaré series. We define for complex variables  $s_1, \dots, s_m$  and  $Z = X + iY \in \mathcal{H}_m$  the non-holomorphic functions

$$H_\tau(Z, s_1, \dots, s_m) = e^{2\pi i \operatorname{tr}(\tau Z)} \cdot \prod_{q=1}^m \operatorname{tr}((\tau^{\frac{1}{2}} Y \tau^{\frac{1}{2}})^{[q]})^{s_q},$$

and the corresponding operator valued non-holomorphic Poincaré series

$$P_\tau(g, s_1, \dots, s_m) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \rho(J(\gamma g, i)^{-\kappa}) \cdot H_\tau(\gamma g \cdot i, s_1, \dots, s_m).$$

Here  $Y^{[q]}$  denotes the  $q$ -th alternating power of the matrix  $Y$ . These Poincaré series converge if  $s_q \gg 0$  for  $q = 1, \dots, m$ . We note this precisely for the special case where we restrict to two variables  $s_1, s_m$ .

**Theorem 2.1.** [5], [8] *Let  $(l_1, \dots, l_m)$ ,  $l_1 \geq \dots \geq l_m = \kappa$  be the dominant highest weight of the representation  $\rho$ . The Poincaré series  $P_\tau(g, s_1, 0, \dots, 0, s_m)$  converge absolutely and uniformly on compact sets with respect to the operator norm in the domain*

$$\left\{ (s_1, s_m) \in \mathbb{C}^2 \mid \operatorname{Re} s_2 > m - \frac{\kappa}{2} \text{ and } \operatorname{Re}(ms_2 + s_1) > m^2 - \frac{\sum_j l_j}{2} \right\}.$$

For fixed such  $(s_1, s_2)$  and for all  $v \in V_\rho$  the functions  $P_\tau(g, s_1, 0, \dots, 0, s_m) \cdot v$  are bounded and belong to  $L^2(\Gamma \backslash G_m) \cap C^\infty(\Gamma \backslash G_m)$ . In particular, in case the weight  $\kappa > 2m$  is large, at the critical point  $(s_1, s_m) = (0, 0)$  the Poincaré series converge absolutely and are holomorphic as functions on  $\Gamma \backslash G_m$ .

Continuing the Poincaré series analytically to the critical point  $(s_1, \dots, s_m) = (0, \dots, 0)$  is a non-trivial problem in case  $m > 1$  (for  $m = 1$  see [3]).

For scalar  $K$ -type  $\rho = \det^\kappa$  we have the following results in case of genus two and small  $\kappa$ . Here a weight  $\kappa$  is called small if  $m < \kappa \leq 2m$ .

**Theorem 2.2.** [5] *Let the genus  $m = 2$  equal two and the weight  $\kappa = 2m = 4$  be four. Then the Poincaré series  $P_\tau(g, s_1, s_2)$  have analytic continuation to the critical point  $(s_1, s_2) = (0, 0)$ . The limits  $P_\tau(g, 0, 0)$  are holomorphic functions in  $L^2(\Gamma \backslash G_2)$ .*

For the proof we study differential operators  $D$  belonging to the center  $\mathfrak{z}_2$  of the universal enveloping algebra of  $\mathfrak{g}_2^\mathbb{C}$  such that in the equation

$$(2) \quad D(P_\tau(g, s_1, s_2)) = \mathcal{P}_\tau(g, s_1, s_2)$$

the auxiliary Poincaré series  $\mathcal{P}_\tau(g, s_1, s_2)$  have better convergency properties. Applying the resolvents of  $D_\pm$  then gives analytic continuations of the Poincaré series as functions in  $L^2(\Gamma \backslash G_2)$ . So we pick up the classical idea by Maass and Selberg for the trace formula. The main difference is that for trace formulas the input functions are Eisenstein series, which naturally are eigenfunctions of  $\mathfrak{z}_2$ , while the non-holomorphic Poincaré series are not.

There are some technically functional analytic arguments to make this work, together with arguments from Langlands' theory of Eisenstein series. We want to mention that there are exactly two differential operators  $D_+$  and  $D_-$  of minimal degree four for which we obtain better convergency properties on the right hand side of (2). On Langland's Eisenstein series  $E_B(g, \Lambda)$  with analytic

spectral parameter  $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$  they have the compelling shape

$$\begin{aligned} D_+(u)E_B(g, \Lambda) &= \prod_{\alpha \text{ long root}} (\check{\alpha}(\Lambda) - u) \cdot E_B(g, \Lambda), \\ D_-(v)E_B(g, \Lambda) &= \prod_{\alpha \text{ short root}} (\check{\alpha}(\Lambda) - v) \cdot E_B(g, \Lambda), \end{aligned}$$

for some affine linear combinations  $u, v$  of the variables  $s_1, s_2$  and weight  $\kappa$ , where  $\check{\alpha}$  is the dual of the root  $\alpha$ .

Theorem 2 implies Theorem 1.1(iv) on holomorphic projection by Sturm's operator above. For the second small weight  $\kappa = m + 1 = 3$  we have jointly with R. Weissauer the following result.

**Theorem 2.3.** [7] *Let the genus  $m = 2$  equal two and the weight  $\kappa = 3$  be three. Then the Poincaré series  $P_{\tau}(g, s_1, s_2)$  have analytic continuation to the critical point  $(s_1, s_2) = (0, 0)$ . The limits  $P_{\tau}(g, 0, 0)$  are  $C^{\infty}$  functions in  $L^2(\Gamma \backslash G_2)$ . The unique non-zero isotypical spectral components are discrete and given by the holomorphic discrete series representation  $\pi_{(2,1)}^{\text{hol}}$  of minimal  $K$ -type  $(3, 3)$ , and a holomorphic but non-discrete series representation  $\pi_1^{\text{hol}}$  of minimal  $K$ -type  $(1, 1)$ .*

We use the same methods as in [5], but employ much deeper insight to the unitary spectrum of  $\text{Sp}_2(\mathbb{R})$ . Two surprising facts occur. First, the analytic continuations exist as a  $C^{\infty}$ -function in  $L^2(\Gamma \backslash G_2)$ , but they are not holomorphic anymore in the critical point  $(s_1, s_2) = (0, 0)$ . The reason for this is the occurrence of the spectral component  $\pi_1^{\text{hol}}$  of  $K$ -type  $(1, 1)$ , which carries the  $K$ -type  $(3, 3)$  (i.e. scalar weight  $\kappa = 3$ ) non-trivially, but in which only functions of  $K$ -type  $(1, 1)$  are holomorphic. Second, the general expectation that the crucial problems will be located within the continuous spectrum (which happens in case  $m = 1$  and  $\kappa = 1$ ) does not hold. In weight  $\kappa = 3$  the continuous spectral components are well-behaved and do not contribute to the continuation. They will indeed make non-trivial contributions in case  $\kappa = 2$ , so this is a problem of weight  $\kappa = m$ .

The articles [5] and [7] exhaust the scalar weight case in genus two. Because genus two in general holds for the pivot case for all higher genera, we expect all these phenomenons and more to occur for any higher genus.

### 3. STURM'S OPERATOR

The statement of Theorem 2.3 that for  $\kappa = m + 1$  the analytic continuation of the Poincaré series is non-holomorphic in the critical point  $(s_1, s_2)$  is surprising and unexpected. It suggests that Sturm's operator fails to realize the holomorphic projection operator in this case. We show that this is indeed the case.

Consider the spectral component  $\pi_1^{\text{hol}}$  which produces the non-holomorphic part of the continued Poincaré series. The generating cusp form  $h \in [\Gamma, 1]_0$  of this representation has minimal  $K$ -type  $(1, 1)$ . (Here we identify representations of

$\mathrm{GL}_m(\mathbb{C})$  with their highest weight.) The only possibility to obtain a function of  $K$ -type  $(3, 3)$  from  $h$  is by application of Maass' shift operator. This is defined by

$$\Delta_+^{[m]}(h)(Z) = (2i)^m (\tau \otimes \det^{\frac{1-m}{2}})(Y^{-1}) \det(\partial_Z) \left( (\tau \otimes \det^{\frac{1-m}{2}})(Y) \cdot h(Z) \right),$$

which for any irreducible rational  $\mathrm{GL}_m(\mathbb{C})$ -representation  $\tau$  sends  $C^\infty(\mathcal{H}_m, V_\tau)$  to  $C^\infty(\mathcal{H}_m, V_{\tau \otimes \det^2})$ .

**Theorem 3.1.** [7], [6] *Let the genus  $m \geq 2$  be arbitrary. Let  $h \in [\Gamma, k]_0$  be a non-zero holomorphic cusp form of weight  $k$ . Then the image of its Maass shift  $\Delta_+^{[m]}(h)$  under Sturm's operator*

$$St_{k+2} \left( \Delta_+^{[m]}(h) \right)$$

*is non-zero if and only if  $k = m - 1$ .*

On the other hand, the Maass shift  $\Delta_+^{[m]}(h)$  is a non-holomorphic function in the spectral component generated by  $h$ . So the holomorphic projection  $S(\Delta_+^{[m]}(h))$  vanishes. Accordingly, Sturm's operator fails to realize the holomorphic projection operator for weight  $\kappa = m + 1$ .

**Theorem 3.2.** [7], [6] *For genus  $m \geq 2$  and  $\kappa = m + 1$  it holds*

$$St_{m+1}(f) = S(f) + Ph(f),$$

*where the phantom term  $Ph(f)$  is non-zero in general, as there is a contribution by Maass shifts from  $[\Gamma, m - 1]$ .*

By virtue of Theorem 3.2, the process of obtaining holomorphic Poincaré series by analytic continuation of analogs of those for larger weight must be expected to fail in general. Because, such holomorphic continuations would make the argument of Section 1 for Sturm's operator work. This reveals once more the fact that the spaces  $[\Gamma, \kappa]_0$  of holomorphic cusp forms are very difficult to describe when  $\kappa$  is small.

For genus two we are more precise, thereby ruling out the possibility that our choice of Poincaré series simply was unlucky.

**Theorem 3.3.** [7] *For genus  $m = 2$  and weight  $\kappa = 3$  let  $p_T$  be the images on the Siegel half space of the limit series  $P_T(\cdot, 0, 0)$ . Then  $p_T$  decompose*

$$p_T = f_T + \Delta_+^{[2]}(h_T)$$

*as sums of holomorphic cusp forms  $f_T \in [\Gamma, 3]_0$  and Maass derivatives  $\Delta_+^{[2]}(h_T)$ , where  $h_T \in [\Gamma, 1]$ . In general,  $f_T$  and  $h_T$  are non-zero. The forms  $h_T$  can be recovered by the anti-holomorphic Maass operator  $\Delta_-^{[2]}$*

$$\Delta_-^{[2]}(p_T) = \frac{3}{4} \cdot h_T.$$

Here  $\Delta_-^{[m]}$  is given by

$$\Delta_-^{[m]}(f) = (2i)^m (\tau \otimes \det(Y)^{\frac{1-m}{2}}) \det(\bar{\partial}_Z) (\det(Y^{-1})^{\frac{1-m}{2}}) f(Z).$$

It is an interesting question how phantom terms decompose in general. Apart from  $\Delta_+^{[m]}(h)$  for  $h \in [\Gamma, \kappa - 1]_0$  coming from the holomorphic representation  $\pi_{m-1}^{hol}$ , there can appear many other terms coming from unitary representations in which the  $K$ -type  $\rho = (\kappa, \dots, \kappa)$  occurs non-trivially. In terms of cusp forms, this could be any  $\Delta^\sigma(h)$  for a vector valued cusp form  $h \in [\Gamma, \tau]_0$  and a differential operator  $\Delta^\sigma$  such that  $\tau + \sigma = \rho$ .

A result in this direction is the notion of Sturm's operator for general  $K$ -type  $\rho$  (see [8]). Let  $\tau$  run through the positive definite half-integral matrices, and let  $f(Z) = \sum_\tau \rho(\tau^{\frac{1}{2}}) a(\tau, \tau^{\frac{1}{2}} Y \tau^{\frac{1}{2}}) \cdot e^{2\pi i \operatorname{tr}(\tau X)}$  be the Fourier expansion of a vector valued  $C^\infty$ -modular form. Then, sending the vector valued Fourier coefficient  $a(\tau, \tau^{\frac{1}{2}} Y \tau^{\frac{1}{2}})$  to the coefficient  $a(\tau)^T$  defined by the vector valued integral

$$\det(\tau)^{-\frac{m+1}{2}} \int_{Y>0} a(\tau, Y)^T \rho(\tau^{\frac{1}{2}}) C(m, \rho)^{-1} \rho(Y) \rho(\tau^{-\frac{1}{2}}) e^{-2\pi i \operatorname{tr}(Y)} \frac{dY_{inv}}{\det(Y)^{\frac{m+1}{2}}}.$$

we define Sturm's operator

$$St_\rho(f)(Z) = \sum_{\tau>0} \rho(\tau^{\frac{1}{2}}) a(\tau) e^{2\pi i \operatorname{tr}(\tau Z)}.$$

Here  $C(m, \rho)$

$$C(m, \rho) = \int_{Y>0} \rho(Y) e^{-4\pi i \operatorname{tr}(Y)} \frac{dY_{inv}}{\det(Y)^{\frac{m+1}{2}}}$$

is an operator-valued integral such that on holomorphic cusp forms Sturm's operator is the identity. It is classically known to be convergent if the dominant highest weight  $(l_1, \dots, l_m)$  of  $\rho$  satisfies  $l_m > \frac{m-1}{2}$  ([2]). In [8] we prove that  $C(m, \rho)$  to be invertible for genus  $m = 2$  as well as for some general classes of representations. In particular, if  $\rho = \det^\kappa$ , then  $C(m, \rho) = c(\kappa, m)$ , and the notions of Sturm's operator coincide. We have the following result on holomorphic projection generalizing Theorem 1.1 to vector valued forms.

**Theorem 3.4.** [8] *Let  $\rho$  be an irreducible representation of  $\mathrm{GL}_m(\mathbb{C})$  of highest weight  $(l_1, \dots, l_m)$  satisfying  $l_m > 2m$ . Then Sturm's operator realizes the holomorphic projection operator.*

But, generalizing Theorems 3.1 and 3.2, for small weight this does not hold true.

**Theorem 3.5.** [8] *Let the genus  $m$  equal two. Let  $\tau$  be an the irreducible representation of  $\mathrm{GL}_2(\mathbb{C})$  of highest weight  $(k+r, k)$  with  $k \geq 1$  and  $r \geq 0$ . Let  $C(m, \rho)$  be invertible. Let  $h \in [\Gamma, \tau]_0$  be a vector valued holomorphic cusp form of weight  $\tau$ . Then the image of its Maass shift  $\Delta_+^{[m]}(h)$  under Sturm's operator*

$$St_{\tau \otimes \det^2} \left( \Delta_+^{[m]}(h) \right)$$

*is non-zero if and only if  $k = 1$ .*

*In case  $\rho$  has highest weight  $(3+r, 3)$  Sturm's operator does not realize the holomorphic projection but produces phantom terms*

$$St_\rho(f) = S(f) + ph(f).$$

#### 4. SPECTRAL POINT OF VIEW

An irreducible representation containing a non-zero holomorphic function  $H$  of weight  $\rho$  is generated by  $H$  and has minimal  $K$ -type  $\rho$ . Most of them are (limit of) holomorphic discrete series representations, but there are others like  $\pi_1^{hol}$  above. In particular, for  $\rho = (m+1, \dots, m+1)$  the underlying representation is the holomorphic discrete series with Harish-Chandra parameter  $\delta = (m, m-1, \dots, 1)$ . In general,  $\rho = (\kappa_1, \dots, \kappa_m)$  with  $\kappa_m \geq m+1$  (and the natural condition  $\kappa_{j+1} \geq \kappa_j$ ) has Harish-Chandra parameter  $(\kappa_1-1, \kappa_2-2, \dots, \kappa_m-m)$  and belongs to the cone given by the holomorphic discrete series within the root space, that is the cone given by the Weyl chamber translated by  $\delta$ , where  $\delta$  equals half the sum of positive roots.

The wall orthogonal to the short root is characterized by the scalar minimal  $K$ -types  $(\kappa, \dots, \kappa)$ , where  $\kappa \geq m+1$ . For those with  $\kappa > m+1$  Sturm's operator realizes the holomorphic projection correctly at least in case  $m=2$  by Theorem 1.1, i.e. its image belongs to the holomorphic discrete series in question. For  $\kappa = m+1$ , belonging to the apex  $\delta$  of the cone, this is wrong by Theorem 3.2. Then, the image of Sturm's operator has a non-zero part in  $\pi_{m-1}^{hol}$ , which has Harish-Chandra parameter  $(m-2, \dots, -1)$  (respectively, a Weyl-conjugate of this). By Theorem 3.5 the same phenomenon occurs for  $\rho = (3+r, 3)$ ,  $r > 0$ , which are the minimal  $K$ -types of the discrete series on the wall of the cone perpendicular to the long root. So they have Harish-Chandra parameters  $(2+r, 1)$ . There Sturm's operator fails. Whereas Sturm's operator acts as holomorphic projection everywhere else on the cone.

We expect that, for higher genera, Sturm's operator will fail to realize the holomorphic projection on all facets of the cone which are not perpendicular to every short root.

The exhaustive results in genus  $m=2$  ([7]) rely on the analysis of the unitary spectrum of  $\mathrm{Sp}_m(\mathbb{R})$ . For results on higher genera an extensive insight to the unitary spectrum of  $\mathrm{Sp}_m(\mathbb{R})$  is necessary. At the moment, the spectral parts parametrized by parameters  $\Lambda$  belonging to the ball of radius  $\|\delta\|$  are little understood in detail. An exact description of the truly arising small  $K$ -types within (unitary) representations of small minimal  $K$ -types, holomorphic as well as non-holomorphic ones, will be a good help for our program. Explicitly, we have to determine those spectral components which are zeros of the differential operators  $D_\pm$  used for analytic continuation of the Poincaré series. Written with respect to Langlands' Eisenstein series, these elements of the center of the universal enveloping Lie algebra themselves depend on the weight  $\rho$  of the Poincaré series. Their zeros are certain affine lines perpendicular to the roots, which get the nearer to the origin the smaller the weight becomes. From the



representation theoretic point of view, this is why things become interesting for small weight.

## 5. PERSPECTIVES

There are several projects resulting from the discussion above which are ongoing work. These include a general theory of analytic continuation of Poincaré series of exponential type for general genus and for vector valued weights. Also a comprehensive atlas of the state of the art of the representation theoretic questions is needed.

Another current project is the case of half-integral weight. Most of our methods apply to them as well. We have some evidence that there should be a positive result on holomorphic projection even in case of weight  $\kappa = m + \frac{1}{2}$ . This would have direct applications to theta correspondences for certain Hilbert modular surfaces (see [1]).

An even more appealing project is to transfer these questions to the dualized setting, the special orthogonal groups  $\mathrm{SO}(2, n)$ . Some parts of the theory have direct counterparts there, for example the concrete differential operators used for analytic continuation, while other parts must be developed, like systems of Poincaré series. It is the case of special orthogonal groups we think of for arithmetic applications of our results.

In the very focus of our work stands the interpretation of the phantom terms arising from Sturm's formula. They occur in an area where analytic, representation theoretic, and arithmetic objects deeply interact with each other. We expect the phantom terms produced by analysis to carry arithmetic impact. In a first step we study their influence within convolution  $L$ -series involving derivatives of Eisenstein series.

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