CASIMIR OPERATORS FOR SYMPLECTIC GROUPS

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Abstract. We give a full set of generators for the center of the universal envelopping Lie algebra of the symplectic group of arbitrary genus. They are of trace type and are given in terms of a basis chosen such that the action on representations of given K-type becomes transparent. We give examples for the latter.

Introduction

The original intension of this work was to understand the action of Casimir operators on automorphic forms for the symplectic group $\mathrm{Sp}_m(\mathbb{R})$ of a broard class or K-types. We achieve partial results. In [3], this problem is done for the standard first Casimir operator C_1. Using the Cartan decomposition of the symplectic lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, one decomposes $C_1 = \mathrm{tr}(E_+E_-) + k$, where $\mathrm{tr}(E_+E_-)$ is a differential operator on the Siegel halfplane depending only on $\mathfrak{p}^{\mathbb{C}}$ and k is some constant coming from $\mathfrak{k}^{\mathbb{C}}$ depending on the K-type only. Surprisingly, an analog for higher Casimir operators (i.e. elements of the center of the universal envelopping algebra) up to the genus m does not exist in literature. Usually Casimir operators are realized with respect to a Cartan subalgebra which evidently is not of any help here.

We use a basis of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ which has pleasing properties: Lie multiplication as well as matrix multiplication is simple and the dual basis (with respect to half the

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trace) is essentially deduced by rearranging. The basis differs from that used in [3] in the $\mathfrak{k}^{\mathbb{C}}$-part.

The starting point of discussion is the common formula ([1], IV. 7)

$$D_r(\Pi) = \sum_{i_1,\ldots,i_r} \mathrm{tr}\,(\Pi(X_{i_1})\cdots\Pi(X_{i_r}))\,X_{i_1}^*\cdots X_{i_r}^* \tag{1}$$

for Casimir elements. Here, X_i runs through a basis of $\mathfrak{g}^{\mathbb{C}}$, the elements X_i^* form its dual with respect to a non-degenerate, Lie invariant bilinear form and Π is a nontrivial finite-dimensional representation of $\mathfrak{g}$. It is well-known that these elements belong to the center $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ of the universal envelopping algebra and are independent of the chosen basis. What is more, an other choice of Π as well as of the bilinear form alters the results by a common multiple constant. We evaluate this formula for the basis mentioned above and the natural representation of $\mathfrak{g}$ to get a set $\{D_2,\ldots,D_{2m}\}$ of m Casimir elements which indeed generates $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$. As examples, we give precise formulae for D_2, D_4.

We apply the result to determine the action of $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ on a representation of K-type $(\lambda,\ldots,\lambda)$ to be given by that of $\mathrm{tr}(E_+E_-),\ldots,\mathrm{tr}((E_+E_-)^m)$. For automorphic forms, the latter are differential operators on the Siegel halfplane and we have recovered a theorem of Maass ([2], §8, p.116).

1. Notation

Let $G = \mathrm{Sp}_m(\mathbb{R})$ be the real symplectic group of genus m and let $\mathfrak{g} = \mathfrak{sp}_m(\mathbb{R})$ be its Lie algebra. We consider the matrix realization of its complexification $\mathfrak{g}^{\mathbb{C}} \subset M_{2m,2m}(\mathbb{C})$ consisting of those g satisfying

$$g'\begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} g = 0.$$

The Cartan decomposition for $\mathfrak{g}$ implies that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$, where $\mathfrak{k}^{\mathbb{C}}$ is given by those matrices satisfying

$$\begin{pmatrix} A & -S \\ S & A \end{pmatrix}, \quad A' = -A, \quad S' = S,$$

and

$$\mathfrak{p}_\pm \;=\; \left\{ \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix}, \quad X' = X \right\}.$$

Let $e_{kl} \in M_{m,m}(\mathbb{C})$ be the elementary matrix having entries $(e_{kl})_{ij} = \delta_{ik}\delta_{jl}$ and let $X^{(kl)} = \frac{1}{2}(e_{kl} + e_{lk})$.

The elements $E_{\pm kl} = E_{\pm lk}$ of $\mathfrak{p}_\pm$ are defined to be those corresponding to $X = X^{(kl)}$, $1 \le k, l \le m$. Then $E_{\pm kl}$, $1 \le k \le l \le m$, form a basis of $\mathfrak{p}_\pm$. For abbreviation, let

$$E_\pm \;=\; (E_{\pm kl})_{kl}$$

be the matrix with matrix valued entries $E_{\pm kl}$. A basis of $\mathfrak{k}^{\mathbb{C}}$ is given by the elements B_{kl}, $1 \le k, l \le m$, corresponding to $A = \frac{1}{2}(e_{kl} - e_{lk})$ and $S = \frac{i}{2}(E_{kl} + e_{lk})$. Let

$$B \;=\; (B_{kl})_{kl}$$

be the matrix with entries B_{kl} and let B^* be its transpose having entries $B^*_{kl} = B_{lk}$. Lie multiplication in $\mathfrak{g}^{\mathbb{C}}$ is easily checked to be given by

$$[E_{+ij}, E_{+kl}] = 0, \qquad [E_{-ij}, E_{-kl}] = 0,$$

$$[E_{+ij}, E_{-kl}] = \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il},$$

$$[B_{ij}, E_{+kl}] = \delta_{jk}E_{+il} + \delta_{jl}E_{+ik},$$

$$[B_{ij}, E_{-kl}] = -\delta_{ik}E_{-jl} - \delta_{il}E_{-jk},$$

$$[B_{ij}, B_{kl}] = \delta_{jk}B_{il} - \delta_{il}B_{kj}.$$

We denote by $\mathcal{B}$ the nondegenerate bilinear form on $\mathfrak{g}^{\mathbb{C}}$ defined by

$$\mathcal{B}(g,h) = \frac{1}{2}\operatorname{tr}(g \cdot h). \tag{2}$$

With respect to $\mathcal{B}$ we get the following dual basis: $E^*_{\pm kl} = \frac{1}{1+\delta_{kl}} E_{\mp kl}$ as well as $B^*_{kl} = B_{lk}$ for all k, l.

2. Casimir elements

In the following, we study words in the matrices E_+, E_-, B and B^*. Let us define some conditions on these words:

(i) E_+ is followed by E_- or B^*.

(ii) E_- is followed by E_+ or B.

(iii) B is followed by E_+ or B.

(iv) B^* is followed by E_- or B^*.

(v) E_+ occurs with the same multiplicity as E_-.

We start with a combinatorial lemma.

Lemma 2.1. *Let $r > 0$ be an integer. Then there are 2^{2r} possibilities to choose a word w of length $2r$ in the matrices E_+, E_-, B and B^* such that the conditions (i) to (v) are satisfied.*

Proof of Lemma 2.1. We make a second claim slightly modifying Lemma 2.1 and prove it along with the lemma itself by induction on r.

Claim: There are 2^{2r} possibilities to choose a word w of length $2r$ in the matrices E_+, E_-, B and B^* such that the conditions (i) to (iv) and

(v') The multiplicity of E_+ is that of E_- enlarged or reduced by one.

are satisfied.

For $r = 1$, the four possible words of the lemma are E_+E_-, E_-E_+, BB and B^*B^*, while the four possibilities of the claim are E_+B^*, E_-B, BE_+ and B^*E_-. Now look at a word w of length $2(r+1)$ satisfying (i)–(v). First, if w ends with E_+E_- or with E_-E_+, then the initial subword of length $2r$ satisfies (i)–(v). And for any of these initial subwords, the ending among E_+E_- and E_-E_+ is unique. This gives 2^{2r} possibilities for w, using the lemma for r. Similarly, we get 2^{2r} possibilities for a word where $E_\pm$ does not occur in the last two letters. If exactly one of the last two letters is $E_\pm$, then we get $2 \cdot 2^{2r}$ possibilities, this time using the claim for r. Altogether these are $2^{2(r+1)}$ possibilities. Similarly, we get the result for the claim, too. □

In the following, we formally take the trace of a word w in the operator valued matrices. For example,

$$\operatorname{tr}(E_+E_-) = \sum_{k,l} E_{+kl}E_{-lk}.$$

Theorem 2.2. *Let $\mathfrak{g} = \mathfrak{sp}_m(\mathbb{R})$ be the Lie algebra of the symplectic group of genus m.*

(a) *The r-th Casimir element is given by*

$$D_{2r} = \sum_w (-1)^{L(w)} \operatorname{tr}(w),$$

where the sum is over all words w of length $2r$ satisfying conditions (i) to (v) above, and $L(w)$ is the number of times E_-B and BE_+ occur isolatedly in w counted cyclicly.

(b) *The center $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ of the universal envelopping algebra of $\mathfrak{g}^{\mathbb{C}}$ is generated by the m Casimir operators $D_2, \ldots, D_{2m}$.*

Here isolated means that E_-B and BE_+ must not hit each other, e.g. $L(E_-BE_+B^*) = 1$ while $L(E_-BBE_+) = 2$. And cyclic means that we have to take into account that the trace is cyclicly invariant, so e.g. $L(E_+E_-BB) = L(E_-BBE_+) = 2$.

Example 2.3. *By Lemma 2.1, we have to sum over the traces of 2^{2r} words w. So the first two Casimirs are*

$$D_2 \;=\; \mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+) + \mathrm{tr}(BB) + \mathrm{tr}(B^*B^*),$$

$$\begin{aligned} D_4 \;&=\; \mathrm{tr}(E_+E_-E_+E_-) + \mathrm{tr}(E_-E_+E_-E_+) + \mathrm{tr}(BBBB) + \mathrm{tr}(B^*B^*B^*B^*) \\ &\quad + \sum_{\zeta\in Z_4} \big(\mathrm{tr}(\zeta(E_+E_-BB)) + \mathrm{tr}(\zeta(E_-E_+B^*B^*)) - \mathrm{tr}(\zeta(E_+B^*E_-B))\big), \end{aligned}$$

where Z_4 is the group of cyclic permutations of four elements.

Proof of Theorem 2.2. (a) We define the following matrices

$$K_1 = \begin{pmatrix} \mathbf{1}_m & i\mathbf{1}_m \\ -i\mathbf{1}_m & \mathbf{1}_m \end{pmatrix}, \qquad K_2 = \begin{pmatrix} \mathbf{1}_m & -i\mathbf{1}_m \\ i\mathbf{1}_m & \mathbf{1}_m \end{pmatrix},$$

$$P_+ = \begin{pmatrix} \mathbf{1}_m & i\mathbf{1}_m \\ i\mathbf{1}_m & -\mathbf{1}_m \end{pmatrix}, \qquad P_- = \begin{pmatrix} \mathbf{1}_m & -i\mathbf{1}_m \\ -i\mathbf{1}_m & -\mathbf{1}_m \end{pmatrix}.$$

Notice that $K_j^2 = K_j$, while $P_\pm^2 = 0$. In the following, we use the abbriviation

$$e_{jk}K_1 \;=\; \begin{pmatrix} e_{jk} & ie_{jk} \\ -ie_{jk} & e_{jk} \end{pmatrix},$$

for the $(m\times m)$-elementary matrix e_{jk} and the $(2m\times 2m)$-matrix K_1 (analogly $e_{jk}K_2$, $e_{jk}P_\pm$). Now we show that D_{2r} in (1) has the claimed shape. A single summand of D_{2r} looks like

$$\mathrm{tr}(X^{(1)}_{j_1j_2}X^{(2)}_{k_1k_2}\cdots X^{(2r-1)}_{j_{2r-1}j_{2r}}X^{(2r)}_{k_{2r-1}k_{2r}})(X^{(1)}_{j_1j_2}\cdots X^{(2r)}_{k_{2r-1}k_{2r}})^*,$$

where $X^{(n)}_{j_nj_{n+1}}$ runs through the basis $E_{\pm jk}$, $1\le j\le k\le m$, B_{jk}, $1\le j,k\le m$. First we examine conditions for a pair $X^{(n)}_{j_1j_2}X^{(n+1)}_{k_1k_2}$ to occur in some summand. To get nice formulae, we sum over all pairs of the same kind. First, let $X^{(n)}_{j_1j_2} = E_{+j_1j_2}$

and $X^{(n+1)}_{k_1k_2} = E_{-k_1k_2}$. Computing the matrix product $E_{+j_1j_2}E_{-k_1k_2}$, taking duals and rearranging summation, we get

$$
\begin{aligned}
(3)\qquad & \sum_{j_1\le j_2;k_1\le k_2} \frac{\operatorname{tr}(\cdots E_{+j_1j_2}E_{-k_1k_2}\cdots)}{(1+\delta_{j_1j_2})(1+\delta_{k_1k_2})}(\cdots)^*E_{-j_1j_2}E_{+k_1k_2}(\cdots)^* \\
&= \frac{1}{4}\sum_{j_1,j_2,k_1,k_2} \operatorname{tr}\big((\delta_{j_1k_1}e_{j_2k_2}+\delta_{j_1k_2}e_{j_2k_1}+\delta_{j_2k_1}e_{j_1k_2}+\delta_{j_2k_2}e_{j_1k_1})K_2\big) \\
&\qquad\qquad\cdot(\cdots)^*E_{-j_1j_2}E_{+k_1k_2}(\cdots)^* \\
&= \sum_{j_1,j_2,k_1} \operatorname{tr}(\cdots e_{j_1k_1}K_2\cdots)(\cdots)^*E_{-j_1j_2}E_{+j_2k_1}(\cdots)^*.
\end{aligned}
$$

Similarly we get for the other choices of basis elements

$$
\begin{aligned}
(4)\qquad & \sum_{j_1\le j_2;k_1\le k_2} \operatorname{tr}(\cdots E_{-j_1j_2}E_{+k_1k_2}\cdots)(\cdots E_{-j_1j_2}E_{+k_1k_2}\cdots)^* \\
&= \sum_{j_1,j_2,k_1} \operatorname{tr}(\cdots e_{j_1k_1}K_1\cdots)(\cdots)^*E_{+j_1j_2}E_{-j_2k_1}(\cdots)^*,
\end{aligned}
$$

$$
\begin{aligned}
(5)\qquad & \sum_{j_1\le j_2;k_1,k_2} \operatorname{tr}(\cdots E_{+j_1j_2}B_{k_1k_2}\cdots)(\cdots E_{+j_1j_2}B_{k_1k_2}\cdots)^* \\
&= -\sum_{j_1,j_2,k_1} \operatorname{tr}(\cdots e_{j_1k_1}P_+\cdots)(\cdots)^*E_{-j_1j_2}B_{j_2k_1}(\cdots)^*,
\end{aligned}
$$

$$
\begin{aligned}
(6)\qquad & \sum_{j_1\le j_2;k_1,k_2} \operatorname{tr}(\cdots E_{-j_1j_2}B_{k_1k_2}\cdots)(\cdots E_{-j_1j_2}B_{k_1k_2}\cdots)^* \\
&= \sum_{j_1,j_2,k_1} \operatorname{tr}(\cdots e_{j_1k_1}P_-\cdots)(\cdots)^*E_{+j_1j_2}B^*_{j_2k_1}(\cdots)^*,
\end{aligned}
$$

$$
\begin{aligned}
(7)\qquad & \sum_{j_1,j_2;k_1\le k_2} \operatorname{tr}(\cdots B_{j_1j_2}E_{-k_1k_2}\cdots)(\cdots B_{j_1j_2}E_{-k_1k_2}\cdots)^* \\
&= -\sum_{j_1,j_2,k_1} \operatorname{tr}(\cdots e_{j_1k_1}P_-\cdots)(\cdots)^*B_{j_1j_2}E_{+j_2k_1}(\cdots)^*,
\end{aligned}
$$

$$\sum_{j_1,j_2;k_1\le k_2} \mathrm{tr}(\cdots B_{j_1j_2}E_{+k_1k_2}\cdots)(\cdots B_{j_1j_2}E_{+k_1k_2}\cdots)^* = \sum_{j_1,j_2,k_1} \mathrm{tr}(\cdots e_{j_1k_1}P_+\cdots)(\cdots)^* B^*_{j_1j_2}E_{-j_2k_1}(\cdots)^*, \tag{8}$$

$$\begin{aligned}\sum_{j_1,j_2,k_1,k_2} &\mathrm{tr}(\cdots B_{j_1j_2}B_{k_1k_2}\cdots)(\cdots B_{j_1j_2}B_{k_1k_2}\cdots)^* \\ &= \sum_{j_1,j_2,k_1} \big[\mathrm{tr}(\cdots e_{j_1k_1}K_1\cdots)(\cdots)^* B_{j_1j_2}B_{j_2k_1}(\cdots)^* \\ &\qquad + \mathrm{tr}(\cdots e_{j_1k_1}K_2\cdots)(\cdots)^* B^*_{j_1j_2}B^*_{j_2k_1}(\cdots)^*\big].\end{aligned} \tag{9}$$

From equations (3) to (9) we get the following conditions for the occuring summands

(i') $E_{+j_1j_2}$ is followed only by $E_{-j_2k_1}$ or $B^*_{j_2k_1}$.

(ii') $E_{-j_1j_2}$ is followed only by $E_{+j_2k_1}$ or $B_{j_2k_1}$.

(iii') $B_{j_1j_2}$ is followed only by $E_{+j_2k_1}$ or $B_{j_2k_1}$.

(iv') $B^*_{j_1j_2}$ is followed only by $E_{-j_2k_1}$ or $B^*_{j_2k_1}$.

These conditions correspond to the former (i) to (iv). Now let us sum over $j_1,\dots,j_{2r}$, $k_1\dots,k_{2r}$.

$$\begin{aligned} &:= \sum_{j_1,\dots,j_{2r},k_1\dots,k_{2r}} \mathrm{tr}(X^{(1)}_{j_1j_2}X^{(2)}_{k_1k_2}\cdots X^{(2r-1)}_{j_{2r-1}j_{2r}}X^{(2r)}_{k_{2r-1}k_{2r}})(X^{(1)}_{j_1j_2}\cdots X^{(2r)}_{k_{2r-1}k_{2r}})^* \\ &= (-1)^{L(w)}\sum_{j_1,\dots,j_{2r},k_1\dots,k_r} \mathrm{tr}(e_{j_1k_1}e_{j_3k_2}\cdots e_{j_{2r-1}k_r}\sigma(w))\tilde X^{(1)}_{j_1j_2}\tilde X^{(2)}_{j_2k_1}\cdots\tilde X^{(2r-1)}_{j_{2r-1}j_{2r}}\tilde X^{(2r)}_{j_{2r}k_r} \\ &= (-1)^{L(w)}\sum_{j_1,\dots,j_{2r}} \mathrm{tr}(e_{j_1j_1}\sigma(w))\tilde X^{(1)}_{j_1j_2}\tilde X^{(2)}_{j_2j_3}\cdots\tilde X^{(2r)}_{j_{2r}j_1},\end{aligned} \tag{10}$$

where $\tilde E_\pm = E_\mp$, $\tilde B = B^*$, $\tilde B^* = B$. Depending only on the word $w = \tilde X^{(1)}\cdots\tilde X^{(2r)}$, $\tilde X^{(l)} \in \{E_+, E_-, B, B^*\}$, there is some sign $(-1)^{L(w)}$ and a matrix $\sigma(w)$ which is a product of r matrices of the form $K_{1/2}, P_\pm$. In this way we get the sum over one type of word w satisfying conditions (i)–(iv). All other words do not occur in D_{2r}.

We evaluate (10) further. First we assume that w is a word in B, B^* only. Then $(-1)^{L(w)} = 1$ and $\sigma(w)$ is a product in K_1 and K_2. As $K_1K_2 = 0 = K_2K_1$, we get

$$\begin{aligned}
(10) \quad &= \sum_{j_1,\ldots,j_{2r}} \left(\mathrm{tr}(e_{j_1j_1}K_1)B_{j_1j_2}\ldots B_{j_{2r}j_1} + \mathrm{tr}(e_{j_1j_1}K_2)B^*_{j_1j_2}\ldots B^*_{j_{2r}j_1}\right)\\
&= \sum_{j_1,\ldots,j_{2r}} \left(B_{j_1j_2}\ldots B_{j_{2r-1}j_{2r}} + B^*_{j_1j_2}\ldots B^*_{j_{2r}j_1}\right)\\
&= \mathrm{tr}(B^{2r}) + \mathrm{tr}((B^*)^{2r}).
\end{aligned}$$

Now we allow $E_\pm$ to occur in w. Then $\sigma(w)$ is a product of $K_{1/2}$ and $P_\pm$. Notice that $P_+P_- = K_2$, $P_-P_+ = K_1$, $P_+K_2 = 0 = K_1P_+$, $P_+K_1 = P_+ = K_2P_+$, $P_-K_1 = 0 = K_2P_-$ and $P_-K_2 = P_- = K_1P_-$. Thus, $\sigma(w)$ does not vanish if and only if w satisfies (i) to (iv). Additionally, $\mathrm{tr}(\sigma(w)) \neq 0$ if and only if P_+ occurs exactly as often as P_-, i.e. if and only if

(v) E_+ occurs with the same multiplicity as E_-.

In this case $\mathrm{tr}(e_{j_1j_1}\sigma(w)) = 1$ and we have

$$(10) = (-1)^{L(w)}\,\mathrm{tr}(w).$$

Thus, part (a) of the theorem is proved apart from the sign $(-1)^{L(w)}$. To compute this sign, we must count the signs (-1) given by equations (5) and (7) in the right way. That is, we find $L(w)$ to be the number of times E_-B and BE_+ occur isolatedly in w cyclicly. For part (b) we notice that as long as $r < m$, the element $D_{2(r+1)}$ is not a polynomial in $D_2, \ldots, D_{2r}$. For example, we never get $\mathrm{tr}((E_+E_-)^{r+1})$ as a combination of $\mathrm{tr}(E_+E_-), \ldots, \mathrm{tr}((E_+E_-)^r)$. On the other hand it is well known and due to the Harish-Chandra isomorphism that $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ is generated by m elements of length $2, \ldots, 2m$. So $D_2, \ldots, D_{2m}$ must do. $\square$

3. Applications

Let us assume we have an admissible representation Π of $G = \mathrm{Sp}_m(\mathbb{R})$ and let us look at its isotypical component Π_ρ for some irreducible representation ρ of the maximal compact subgroup $K = K_m$. As K is isomorphic to the unitary group U_m by

$$J : K \to U_m, \qquad \begin{pmatrix} A & -S \\ S & A \end{pmatrix} \mapsto A + iS,$$

ρ is characterized by its highest weight $(\lambda_1, \dots, \lambda_m)$. Let $v_h \neq 0$ be a highest weight vector of ρ. For $j \geq k$, the action of B_{jk} on v_h is determined by

$$\begin{aligned} \rho(B_{jk})v_h &= \frac{d}{dt}\rho(\exp(tJ(B_{jk})))v_h \mid_{t=0} \\ &= \begin{cases} \frac{d}{dt}\rho(diag(1,\dots,e^{-t},1,\dots 1))v_h \mid_{t=0} = -\lambda_j v_h, & \text{for } j = k, \\ \frac{d}{dt}\rho(\mathbf{1} - te_{kj})v_h \mid_{t=0} = 0, & \text{for } j > k, \end{cases} \end{aligned}$$

as $\exp(tJ(B_{jk}))$ is an upper triangular matrix if $j \leq k$. Similarly we get for a lowest weight vector v_l,

$$\rho(B_{kj})v_l = \begin{cases} -\lambda_j v_l, & \text{for } j = k, \\ 0, & \text{for } j > k. \end{cases}$$

Next we notice that for all words w occuring in Theorem 2.2, $\mathrm{tr}(w)$ is $\mathfrak{k}^{\mathbb{C}}$-invariant (as we get telescopic sums for the commutators). Thus by Schur's lemma, the Casimirs' action on Π_ρ is deduced by the actions of their single summands $\mathrm{tr}(w)$ on each K-irreducible component. On these components the summands are constant given by evaluation on the highest weight vector, for example.

Furthermore, $\mathrm{tr}(B^{2r})$, $\mathrm{tr}((B^*)^{2r})$ belong to $\mathfrak{Z}(\mathfrak{k}^{\mathbb{C}})$, so they act by constants on Π_ρ deducible by $\rho(B_{jk})v_h$, $j \geq k$. For example, if we rearrange

$$\mathrm{tr}(BB) = \mathrm{tr}(B^*B^*) = \sum_j B_{jj}^2 + \sum_{k<j}\big(2B_{kj}B_{jk} + B_{jj} - B_{kk}\big),$$

then we get

$$\rho(\mathrm{tr}(BB))v_h = \sum_j \bigl(\lambda_j^2 + (m+1-2j)\lambda_j\bigr)v_h.$$

For the general case notice that in any summand $B_{j_1j_2}\dots B_{j_{2r}j_1}$ of $\mathrm{tr}(B^{2r})$ there is some $B_{j_nj_{n+1}}$ where $j_n > j_{n+1}$, if not all j_n are equal. So by rearranging, we can determine the action of this summand by the action of terms of lower length.

For words w in which both $E_\pm$ and B, B^* occur, the evaluation of $\mathrm{tr}(w)$ is not that simple. But rearranging $\mathrm{tr}(w)$ (thereby producing terms of lower length satisfying again conditions (i)–(v) above) such that all terms B, B^* are collected on the right, they can be evaluated first. For example, for the first two Casimirs (see Ex. 2.3) we get

Corollary 3.1. *Let $C_1 := \frac{1}{2}D_2$ and $C_2 := \frac{1}{2}D_4$. Then*

$$C_1 = \frac{1}{2}(\mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+)) + \mathrm{tr}(BB),$$

$$\begin{aligned} C_2 &= \frac{1}{2}\bigl(\mathrm{tr}(E_+E_-E_+E_-) + \mathrm{tr}(E_-E_+E_-E_+) + \mathrm{tr}(B^4) + \mathrm{tr}((B^*)^4)\bigr) \\ &\quad +2\bigl(\mathrm{tr}(E_+E_-BB) + \mathrm{tr}(E_-E_+B^*B^*)\bigr) - \sum_{i,j,k,l}\{(E_+)_{kl}, (E_-)_{ij}\}B_{jk}B_{il} \\ &\quad +\frac{(m+1)^2}{2}(\mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+)), \end{aligned}$$

where

$$\frac{1}{2}(\mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+)) = \mathrm{tr}(E_+E_-) - (m+1)\,\mathrm{tr}(B),$$

$$\begin{aligned} \frac{1}{2}(\mathrm{tr}(E_+E_-E_+E_-) + \mathrm{tr}(E_-E_+E_-E_+)) &= \mathrm{tr}(E_+E_-E_+E_-) \\ -\frac{1}{2}(\mathrm{tr}(E_+E_-) + \mathrm{tr}(E_-E_+))\,\mathrm{tr}(B) &- \frac{m+2}{2}\bigl(\mathrm{tr}(E_+E_-B) + \mathrm{tr}(E_-E_+B^*)\bigr). \end{aligned}$$

In the case $\rho = (\lambda, \dots, \lambda)$, we now have transparent formulae at hand. Here ρ has dimension one, so highest and lowest weight vectors coincide and we have $\rho(B_{jk}) =$

$-\lambda\delta_{jk}$. So the only components of B, B^* left are B_{jj} which produce a common constant. For example,

$$\Pi_\rho(C_1) = \Pi_\rho(\mathrm{tr}(E_+E_-)) + \lambda m(m+1+\lambda)$$

and

$$\begin{aligned}\Pi_\rho(C_2) &= \Pi_\rho(\mathrm{tr}(E_+E_-E_+E_-)) + m\lambda^4 \\ &\quad +((m+1)^2 + 2\lambda(m+1) + 2\lambda^2)\big(\Pi_\rho(\mathrm{tr}(E_+E_-) + \lambda m(m+1)\big).\end{aligned}$$

Similarly it is evident that

$$\Pi_\rho(D_{2r}) \;=\; 2\Pi_\rho(\mathrm{tr}((E_+E_-)^r)) + \Pi_\rho(P_{2r}),$$

where P_{2r} is a polynomial in $\mathrm{tr}(E_+E_-), \ldots, \mathrm{tr}((E_+E_-)^{r-1})$. So we get

Corollary 3.2. *On Π_ρ, $\rho = (\lambda, \ldots, \lambda)$, the Casimir operators are exactly the polynomials in* $\mathrm{tr}(E_+E_-), \ldots, \mathrm{tr}((E_+E_-)^m)$.

As an application, we consider modular forms on the Siegel halfplane $\mathcal{H}_m$. For an irreducible representation (V, ρ) of $\mathrm{GL}_m(\mathbb{C})$ (equivalently of K_m), let $f : \mathcal{H}_m \to V$ be a C^∞-function of moderate growth satisfying

$$f(g.z) \;=\; \rho(cz+d)f(z)$$

for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_m(\mathbb{Z})$, $g.z = (az+b)(cz+d)^{-1}$. That is, f is a non-holomorphic modular form for ρ. Then $f(g) = \rho^*(ci+d)f(g.i\mathbf{1}_m)$ defines an automorphic form on G. If more precisely f is a modular form of weight κ, then $\rho^* = (-\kappa, \ldots, -\kappa)$. By Corollary 3.2, the action of $\mathfrak{Z}(\mathfrak{g}^{\mathbb{C}})$ on such modular forms is given by evaluating $\mathrm{tr}(E_+E_-), \ldots, \mathrm{tr}((E_+E_-)^m)$, which are differential operators on $\mathcal{H}_m$ ([3], Ch. 3, 4).

Especially, if f is holomorphic, then $\mathrm{tr}((E_+E_-)^r)f = 0$. We achieved the differential operators given in [2] (§8, p.116) by purely algebraical means.

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