SUPER CATALAN NUMBERS AND THE EULER OPERATOR

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Abstract. We show that super Catalan numbers (defined by Gessel) occur as constant coefficients of polynomials arising as solutions of combinatorial polynomials under a summation operator. We give new sum representations for these super Catalan numbers.

1. Introduction and statement of result

Super Catalan numbers

$$C(n,k) \;=\; \frac{(2n)!\cdot(2k)!}{2\cdot n!\cdot k!\cdot(n+k)!}$$

for integers $n,k\geq 0$ were introduced By Gessel [1] as special super ballot numbers. In particular, $C(n,1)$ are the Catalan numbers. Notice that in contrast to [1] we normalize by a factor $\frac{1}{2}$. Super Catalan numbers are integers apart from $C(0,0)$, symmetric, and satisfy the summation equation [1, p. 191]

$$C(n+1,k)+C(n,k+1) \;=\; 4\cdot C(n,k)\,. \tag{1}$$

We came along these numbers during our study of polynomial solutions F of the summation operator

$$\big(SF\big)(x) \;=\; F(x+1)+F(x) \;=\; f(x)\,.$$

Choosing $f(x,\nu,n) = \begin{bmatrix} x \\ n \end{bmatrix}\begin{bmatrix} x-\nu \\ n \end{bmatrix}$ on the right hand side, the constant terms of the solutions $F(x,\nu,n)$ are either zero or equal to $(-1)^{\mu}2^{-2n}C(n,\mu)$ for some μ depending on ν, see Proposition 3.1.

Let us denote by E the Euler operator, i.e. the inverse of the summation operator S on polynomials. Then E is uniquely determined by the recursion, $E(x^0)=\frac{1}{2}$, and for $n>0$

$$E(x^n) \;=\; \frac{1}{2}x^n - \frac{1}{2}\sum_{j=0}^{n-1}\binom{n}{j}E(x^j)\,.$$

The polynomials $e_n(x)=E(x^n)$ are the well-known Euler polynomials defined by the generating series

$$\sum_{n=0}^{\infty} e_n(x)\frac{t^n}{n!} \;=\; \frac{e^{xt}}{e^t+1}\,.$$

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Because $S\big(\frac{e^{xt}}{e^t+1}\big) = e^{xt}$ the polynomials indeed satisfy $Se_n(x) = x^n$. In particular, $e_1(x) = \frac{1}{2}x - \frac{1}{4}$, $e_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x$, $e_3(x) = \frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{1}{8}$, and $e_4(x) = \frac{1}{2}x^4 - x^3 + \frac{1}{2}x$. The values $E_{2m} = 2^{2m+1}e_{2m}(\frac{1}{2})$ are the alternating Euler numbers, $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$, whereas for all $m \geq 0$ one has $e_{2m+1}(-\frac{1}{2}) = 0$. From this point of view, super Catalan numbers play a role for the polynomials $f(x,\nu,n)$ similar to that of the Euler numbers for the polynomials x^n.

The solutions $F(x,\nu,n)$ of the summation equation in turn were part of our study on dimensions of Lie super modules [2], [3]. The nomenclature super Catalan numbers happens to be a very lucky one allowing this ambiguity.

As a consequence of different presentations of the constant terms of the solution polynomials $F(x,\nu,n)$ above we find the following sum representations for super Catalan numbers.

Theorem 1.1. *The super Catalan numbers $C(n,\mu)$ satisfy the following identities.*

(a) *For all $0 \leq \mu \leq n$*

$$2 \cdot C(n,0) \ = \ \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \binom{k}{n} \binom{k-\mu}{n} .$$

(b) *For all $0 \leq \mu \leq \lfloor \frac{n-1}{2} \rfloor$*

$$C(n,0) \ = \ \sum_{k=0}^{2n} (-1)^{k+1} \binom{k}{n} \binom{k+1+2\mu-n}{n} \sum_{j=k+1}^{2n} \binom{2n}{j} ,$$

whereas for all $0 \leq \mu \leq \lfloor \frac{n}{2} \rfloor$

$$(-1)^\mu \cdot C(n-\mu,\mu) - C(n,0) \ = \ \sum_{k=0}^{2n} (-1)^k \binom{k}{n} \binom{k+2\mu-n}{n} \sum_{j=k+1}^{2n} \binom{2n}{j} .$$

Notice that in each sum, the summands actually are zero for $k < n$.

For the proof it is important to recognize the super Catalan numbers to satisfy a second recursion formula

$$C(n,k) \ = \ C(n-1,k+1) + \frac{4(n-k-1)}{n+k} \cdot C(n-1,k) ,$$

for all $n > 0$ and all $0 \leq k \leq n$.
The solutions $F(x,\nu,n)$ are determined by a general formula for the polynomial preimage $E(f)$ of a polynomial f given by the values $f(x_j)$ for subsequent $x_j = a + jt$ for $j = 0, \ldots, \deg(f)$ and constants a and t. We develop this method in section 2 and give the proof of theorem 1.1 in section 3.

2. Euler operator

We give an intrinsic description of the preimage $E(f)$ of a polynomial f of degree n, in case we know the values $f(x_j)$ at subsequent places $x_j = a + jt$ for $j = 0, \ldots, n$ and constants a and $t > 0$. Without loss of generality, we assume $a = 0$ and $t = 1$. Define a sequence F_k, $k \in \mathbb{N}_0$, by $F_0 = 0$, and for all $k \geq 0$

$$F_{k+1} \;=\; f(k) - F_k\,.$$

The sequence F_k is a solution of the sequence of discrete equations

$$SF_k \;=\; f(k)\,.$$

Any other solution differs from F_k only by a sequence $(-1)^k \cdot c$ for some constant c. Let $G(x)$ be the interpolation polynomial of degree at most $n = \deg f$ of the values $G(k) = F_k$ for $k = 0, \ldots, n$. It is given by Lagrange interpolation

$$G(x) \;=\; \sum_{j=0}^{n} F_j \cdot \prod_{k=0,k\neq j}^{n} \frac{x-k}{j-k} \;=\; \sum_{j=0}^{n} \frac{(-1)^{n-j}F_j}{j!(n-j)!} \prod_{k=0,k\neq j}^{n} (x-k)$$

and satisfies the summation equation

$$G(k+1) + G(k) \;=\; f(k)$$

for $k = 0, 1, \ldots n-1$. In order to obtain the polynomial solution $F(x) = E(f(x))$, let $B_n(x)$ be the polynomial of degree n which interpolates the values $B_n(k) = (-1)^k$ for $k = 0, 1, \ldots, n$. Define

$$F(x) \;=\; G(x) + c \cdot B_n(x)\,, \tag{2}$$

where the constant c is determined by the summation equation at $x = n$

$$F(n+1) + F(n) = G(n+1) + G(n) + c \cdot (B_n(n+1) + B_n(n)) \;=\; f(n)\,.$$

Because the polynomial $B_n(x)$ is given explicitly in Lagrange form

$$B_n(x) \;=\; (-1)^n \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \prod_{k=0,k\neq j}^{n} (x-k)\,,$$

it follows

$$B_n(n+1) + B_n(n) \;=\; (-1)^n\big((2^{n+1}-1)+1\big) \;=\; (-1)^n 2^{n+1}\,.$$

We obtain by using $F_{n+1} = \sum_{k=0}^{n} (-1)^{n-k} f(k)$

$$\begin{aligned}(-1)^n 2^{n+1} \cdot c &= f(n) - G(n) - G(n+1)\\ &= (-1)^n \sum_{j=0}^{n} \Big((-1)^j f(j) + \binom{n+1}{j} \sum_{k=0}^{j-1} (-1)^k f(k) \Big)\,,\end{aligned}$$

or equivalently

$$c \;=\; \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k f(k) \sum_{j=k+1}^{n+1} \binom{n+1}{j}\,. \tag{3}$$

We summarize.

Proposition 2.1. *The preimage $F = E(f)$ of the polynomial f of degree n under the summation operator S is the polynomial*

$$F(x) \;=\; (-1)^n \sum_{j=0}^{n} \frac{\big(c+(-1)^j F_j\big)}{j!(n-j)!} \prod_{k=0,k\neq j}^{n} (x-k)\;,$$

with constant $c = F(0)$ given by (3), and coefficients $F_j \;=\; \sum_{k=0}^{j-1}(-1)^{j-1-k}f(k)$.

Multiplying F_j by $1 = 2^{-(n+1)}\sum_{i=0}^{n+1}\binom{n+1}{i}$ we obtain the following expression for the coefficients of the solution polynomial

$$c+(-1)^jF_j \;=\; \frac{1}{2^{n+1}}\left(\sum_{l=j}^{n}(-1)^l f(l)\sum_{i=l+1}^{n+1}\binom{n+1}{i} - \sum_{l=0}^{j-1}(-1)^l f(l)\sum_{i=0}^{l}\binom{n+1}{i}\right).$$

Writing

$$f(x) \;=\; a_n x^n + \cdots + a_1 x + a_0\;,$$

for the solution polynomial it follows $F(x) \;=\; \frac{a_n}{2}x^n + \ldots$. Comparing this with the highest coefficient of F in proposition 2.1 we obtain

$$\frac{a_n\cdot n!}{2} \;=\; (-1)^n 2^n\cdot c + (-1)^n\sum_{j=0}^{n}\binom{n}{j}(-1)^j F_j\;,$$

which is equivalent to a second formula for the constant c

$$c \;=\; \frac{(-1)^n n!}{2^{n+1}}\cdot a_n + \frac{1}{2^n}\sum_{k=0}^{n}(-1)^k f(k)\sum_{j=k+1}^{n}\binom{n}{j}\,. \tag{4}$$

Simplifying the identity (3) = (4) yields the well-known expression for the leading coefficient a_n of the polynomial f

$$(-1)^n n!\cdot a_n \;=\; \sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\,. \tag{5}$$

By this and $c = F(0)$ being the constant coefficient of F, a number of non-obvious combinatorial identities arise. We begin with two simple examples.

Example 2.2. Let $f(x) = x^n$. For the Euler polynomials $e_n(x) = E(x^n)$ we obtain

$$e_n(0) \;=\; c_n \;=\; \frac{1}{2^{n+1}}\sum_{l=0}^{n}(-1)^l l^n\sum_{i=l+1}^{n+1}\binom{n+1}{i}\,.$$

The identities for the leading coefficients $\frac{1}{2} = \sum_{j=0}^{n}\frac{c_n+(-1)^jF_{n,j}}{j!(n-j)!}$ yield for all n

$$(-1)^n 2^n n! \;=\; \sum_{j=0}^{n}\binom{n}{j}\left(\sum_{l=j}^{n}(-1)^l l^n\sum_{i=l+1}^{n+1}\binom{n+1}{i} - \sum_{l=0}^{j-1}(-1)^l l^n\sum_{i=0}^{l}\binom{n+1}{i}\right).$$

Example 2.3. Let

$$f(x) \;=\; \begin{bmatrix} x \\ n \end{bmatrix} \;=\; \frac{1}{n!} \cdot x\bigl(x-1\bigr)\cdots\bigl(x-(n-1)\bigr)\,.$$

The coefficients F_j of the polynomial solution F of the summation equation $SF(x) = f(x)$ given by proposition 2.1 are all zero, whereas $c \;=\; \frac{(-1)^n}{2^{n+1}}$. It follows

$$F(x) \;=\; c \cdot B_n(x) \;=\; \frac{1}{2^{n+1} n!} \sum_{j=0}^{n} \binom{n}{j} \prod_{k=0, k\neq j}^{n} (x-k)\,.$$

3. Proof of Theorem 1.1

Proposition 3.1. *(a) For $\nu = 0, 1, \ldots, n$ define polynomials*

$$f(x,\nu,n) \;=\; \begin{bmatrix} x \\ n \end{bmatrix} \begin{bmatrix} x-\nu \\ n \end{bmatrix}$$

as well as constants

$$c(\nu,n) \;=\; \begin{cases} 0\,, & \text{if } \nu = n-1-2\mu \\ \frac{(-1)^\mu}{2^{2n}} \cdot C(n-\mu,\mu)\,, & \text{if } \nu = n-2\mu \end{cases}\,. \tag{6}$$

The polynomial solution of the summation equation $SF(x,\nu,n) = f(x,\nu,n)$ is

$$F(x,\nu,n) \;=\; \sum_{j=0}^{2n} \bigl(F_j(\nu,n) + (-1)^j c(\nu,n)\bigr) \prod_{k=0,k\neq j}^{2n} \frac{x-k}{j-k}\,,$$

where

$$F_j(\nu,n) \;=\; (-1)^{j-1} \sum_{k=0}^{j-1} (-1)^k \binom{k}{n} \binom{k-\nu}{n}\,.$$

The constants $c(\nu,n)$ satisfy the following two recursion formulas for $2 \le \nu \le n$

$$c(\nu-2,n) \;=\; c(\nu,n) - c(\nu-1,n-1)\,, \tag{7}$$

and

$$c(\nu-2,n) \;=\; -c(\nu,n) + \frac{\nu-1}{n} c(\nu-1,n-1)\,. \tag{8}$$

(b) For

$$f(x,n+1,n) \;=\; \begin{bmatrix} x \\ n \end{bmatrix} \begin{bmatrix} x-n-1 \\ n \end{bmatrix}$$

the polynomial solution of the equation $SF(x,n+1,n) = f(x,n+1,n)$ is

$$F(x,n+1,n) \;=\; \frac{1}{2(2n)!} \left(\sum_{j=0}^{n} \binom{2n}{j} \prod_{k=0,k\neq j}^{2n} (x-k) - \sum_{j=n+1}^{2n} \binom{2n}{j} \prod_{k=0,k\neq j}^{2n} (x-k) \right)\,.$$

In particular we obtain by Example 2.3

$$F(x,n,n) \;=\; \frac{C(n,0)}{2^{2n}}\cdot B_{2n}(x)\;,$$

whereas

$$F(x,n-1,n) \;=\; C(n,0)\cdot \begin{bmatrix} x\\ 2n\end{bmatrix}\;.$$

We emphasize that the values of $F(x,\nu,n)$ at $x=k$ for $0\le k\le 2n$ are explicitly given by

$$F(k,\nu,n) \;=\; (-1)^k c(\nu,n) + \sum_{j=n+\nu}^{k-1}(-1)^{k-1+j}\binom{j}{n}\binom{j-\nu}{n}\;.$$

In particular, for $\nu=n-1-2\mu$ the polynomial $F(x,\nu,n)$ has zeros in $x=0,1,\ldots,n-\nu$. Our main result theorem 1.1 on super Catalan numbers now is a corollary of proposition 3.1.

Proof of theorem 1.1. Part (a) is given by identity (5) for the leading coefficients of the polynomials $F(x,\mu,n)$. Part (b) is given by formula (4) for the constant coefficients $c(\nu,n)$ of $F(x,\nu,n)$ which are also determined by proposition 3.1. Part (b) follows in particular from equation (11) below by inserting the special values of $c(\nu-2,n)$ given in proposition 3.1. □

Proof of proposition 3.1. Part (a): Notice that

$$f(x,\nu,n) \;=\; \frac{1}{n!^2}\prod_{j=0}^{\nu-1}(x-j)(x-n-j)\cdot\prod_{k=1}^{n-\nu}(x-(\nu-1)-k)^2\;.$$

Hence, in proposition 2.1, the series $F_j=F_j(\nu,n)$ is given by

$$F_j(\nu,n) \;=\; (-1)^{j-1}\sum_{k=0}^{j-1}(-1)^k\binom{k}{n}\binom{k-\nu}{n}\;,$$

where the summands vanish for $k<n+\nu$. In particular, $F_j(\nu,n)=0$ for $0\le j\le n+\nu$. We obtain the polynomial solution

$$F(x,\nu,n) \;=\; \sum_{j=0}^{2n}\big(F_j(\nu,n)+(-1)^j c(\nu,n)\big)\prod_{k\neq j}\frac{x-k}{j-k}\;,$$

where the constant $c(\nu,n)$ is determined by (4)

$$c(\nu,n) \;=\; \frac{(2n)!}{2^{2n+1}n!^2} + \frac{1}{2^{2n}}\sum_{k=n+\nu}^{2n}(-1)^k\binom{k}{n}\binom{k-\nu}{n}\sum_{j=k+1}^{2n}\binom{2n}{j}\;. \tag{9}$$

We show that these numbers coincide with those of definition (6). Evaluating (9) we obtain

$$c(n,n) \;=\; \frac{1}{2^{2n}}\cdot C(n,0) \quad\text{ and }\quad c(n-1,n) \;=\; 0\;.$$

We use these special values to prove (6) by increasing induction on n and decreasing induction on ν using recursion formula (8). But first observe that recursion formula (7) follows from recursion formula (1) for super Catalan numbers once the explicit values (6) hold true. For the above induction we have to prove (8). Observe that

$$\binom{k}{n}\binom{k-\nu+2}{n} = \binom{k+1}{n}\binom{k+1-\nu}{n} + \frac{\nu-1}{n}\binom{k}{n-1}\binom{k-(\nu-1)}{n-1}. \tag{10}$$

Using the definition of the super Catalan number, by (9) we know

$$2^{2n}c(\nu-2,n) = C(n,0) + \sum_{k=n+\nu-2}^{2n}(-1)^k\binom{k}{n}\binom{k-\nu+2}{n}\sum_{j=k+1}^{2n}\binom{2n}{j}. \tag{11}$$

We split this expression according to the identity for binomial coefficients (10). For the first part we obtain

$$\begin{aligned}
& C(n,0) + \sum_{k=n+\nu-2}^{2n}(-1)^k\binom{k+1}{n}\binom{k+1-\nu}{n}\sum_{j=k+1}^{2n}\binom{2n}{j} \\
= \;& C(n,0) - \sum_{k=n+\nu}^{2n+1}(-1)^k\binom{k}{n}\binom{k-\nu}{n}\sum_{j=k}^{2n}\binom{2n}{j}.
\end{aligned}$$

Using formula (5) for the highest coefficient $(n!)^{-2}$ of the polynomial $f(x,\nu,n)$ to sum the terms with $j=k$, this equals

$$-C(n,0) - \sum_{k=n+\nu}^{2n}(-1)^k\binom{k}{n}\binom{k-\nu}{n}\sum_{j=k+1}^{2n}\binom{2n}{j} = (-1)\cdot 2^{2n}\cdot c(\nu,n),$$

where the last equality is due to (4). For the second part of (11) we obtain

$$\begin{aligned}
& \sum_{k=n+\nu-2}^{2n}(-1)^k\binom{k}{n-1}\binom{k-(\nu-1)}{n-1}\sum_{j=k+1}^{2n}\binom{2n}{j} \\
= \;& 2\cdot\sum_{k=n-1+(\nu-1)}^{2(n-1)}(-1)^k\binom{k}{n-1}\binom{k-(\nu-1)}{n-1}\sum_{j=k+1}^{2n-1}\binom{2n-1}{j} \\
& + \sum_{k=n-1+(\nu-1)}^{2n}\binom{2n-1}{k}(-1)^k\binom{k}{n-1}\binom{k-(\nu-1)}{n-1},
\end{aligned}$$

which equals

$$2^{2n}\cdot c(\nu-1,n-1)+2\cdot C(n-1,0)+\sum_{k=n-1+(\nu-1)}^{2n-1}\binom{2(n-1)}{k-1}(-1)^k\binom{k}{n-1}\binom{k-(\nu-1)}{n-1}.$$

Using that the last sum

$$\sum_{k=n-1+(\nu-1)}^{2n-1} \binom{2(n-1)}{k-1}(-1)^k \binom{k}{n-1}\binom{k-(\nu-1)}{n-1}$$
$$= \; -\sum_{k=0}^{2(n-1)} \binom{2(n-1)}{k}(-1)^k g(k) \;=\; -2\cdot C(n-1,0)$$

equals up to the constant $(2(n-1))!$ the leading coefficient of the polynomial $g(x) = f(x+1,\nu-1,n-1)$, the second part of (11) becomes $2^{2n}\cdot c(\nu-1,n-1)$. So putting the two parts together, we obtain for (11)

$$2^{2n}c(\nu-2,n) \;=\; (-1)\cdot 2^{2n}c(\nu,n) + \frac{2^{2n}\cdot(\nu-1)}{n}\cdot c(\nu-1,n-1)\,.$$

We have proved recursion formula (8). In order to finish the induction argument, by hypothesis we assume that (6) holds true for $c(\nu,n)$ as well as for $c(\nu-1,n-1)$. If ν is of the form $\nu = n-1-2\mu$ these two constants are zero, so (8) implies $c(\nu-2,n) = 0$. If $\nu = n-2\mu$ we obtain for the right hand side of (8)

$$\frac{(-1)^{\mu+1}}{2^{2n}}\Big(C(n-\mu,\mu) - \frac{n-2\mu-1}{n}\cdot 4\cdot C(n-1-\mu,\mu)\Big) \;=\; \frac{(-1)^{\mu+1}}{2^{2n}}C(n-(\mu+1),\mu+1)\,,$$

which must equal the left hand side $c(n-2(\mu+1),n)$ of (8).

Part (b): We proceed again by proposition 2.1 to obtain the values

$$F_j(n+1,n) \;=\; \sum_{k=0}^{j-1}(-1)^{j-1-k}f(k,n+1,n) \;=\; \begin{cases} 0 & \text{if } j=0,\dots,n \\ (-1)^{j-1} & \text{if } j=n+1,\dots,2n \end{cases}\,,$$

as well as the constant

$$c = 2^{-2n}\Big(\frac{(2n)!}{2n!^2} + \sum_{k=0}^{2n}(-1)^k f(k,n+1,n)\sum_{j=k+1}^{2n}\binom{2n}{j}\Big)$$
$$= 2^{-2n}\Big(\frac{1}{2}\binom{2n}{n} + \sum_{j=n+1}^{2n}\binom{2n}{j}\Big) \;=\; \frac{1}{2}\,.$$

This leads to the solution polynomial

$$F(x,n+1,n) = \frac{1}{2(2n)!}\left(\sum_{j=0}^{n}\binom{2n}{j}\prod_{k=0,k\neq j}^{2n}(x-k) - \sum_{j=n+1}^{2n}\binom{2n}{j}\prod_{k=0,k\neq j}^{2n}(x-k)\right)\,.$$

□

References

[1] Gessel I. M.: *Super Ballot Numbers,* J. of Symbolic Computation Vol. 14, 2-3, 1992, 179-194.

[2] Maurischat K., Weissauer R.: *Summation operators and a family of polynomials related to dimensions of $Sl(n|n)$-modules,* preprint 2019

[3] Maurischat K., Weissauer R.: *The dimensions of irreducible $\mathfrak{gl}(3,3)$ and $\mathfrak{gl}(4,4)$ modules,* preprint 2019.

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