SUMMATION OPERATORS AND A FAMILY OF POLYNOMIALS RELATED TO DIMENSIONS OF $Sl(n|n)$-MODULES

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Abstract. We investigate several families of polynomials which are related by summation operators. They satisfy interesting combinatorial properties like being integer-valued at integral points. This involves nearby-symmetries and a recursion for the values at half-integral points. We obtain identities interpolating the Delannoy numbers. The results are motivated by our study of irreducible Lie supermodules, where some of the polynomial families define dimensions.

Contents

1. Introduction

We study a family of polynomials that arises from dimension formulas for irreducible finite dimensional maximal atypical representations of the superlinear groups $Sl(n|n)$. So far explicit dimension formulas in this context have been unknown even in special cases. In this paper we discuss a series of examples. It is instructive to compare the situation with the classical formulas for irreducible representations of the group $Sl(n)$.

Classical case. The isomorphism classes of finite dimensional irreducible representations of the special linear group $Sl(n)$ over $\mathbb{C}$ are uniquely described by their dominant weights λ. These λ are parametrized by the integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0$ such that the dimensions of the corresponding irreducible representations depend on the λ_i is a polynomial way. As an example, consider the symmetric powers $S^k(\mathbb{C}^n)$ of the n-dimensional standard representation $\mathbb{C}^n$ of $Sl(n)$. They are irreducible with dominant weight corresponding to $\lambda_1 = k$ and $\lambda_i = 0$ for $i \geq 2$. Their dimension is the

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dimension of the vectorspace of homogeneous polynomials of degree k in n variables, which is $\binom{n+k-1}{k-1}$ by induction on n. Using an index shift we define

$$P_{cl}(n,m) := \dim(S^{m-1}(\mathbb{C}^n)) \ ,$$

so that the number $P_{cl}(n,m)$ becomes symmetric in n and m and polynomial in the variable m of degree $n-1$

$$P_{cl}(n,m) = \binom{n+m-2}{m-1} = \binom{m+n-2}{n-1} \ .$$

Superlinear case. For the special linear supergroup $Sl(n|n)$ the isomorphism classes of finite dimensional irreducible representations are again described by dominant weights ([5], [8]). Those dominant weights belonging to maximal atypical representations are again parametrized by the integers $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_{n-1} \geq 0$ in a natural way. Although there exists a combinatorial character formula [8], [2], the so called atypical irreducible representations [5], and among these in particular those of maximal atypicality, are not entirely understood. In particular, no counterpart of the Weyl dimension formula is known. Worse than that, in general the dimensions of these representations do not depend on the coefficients λ_i in a polynomial way [6]. In the simple cases $\lambda_1 = k \geq 0$ and $\lambda_i = 0$ for $i \geq 2$ as before, let denote S^k the irreducible maximal atypical representations of $Sl(n|n)$ associated to these dominant weights. Notice, in contrast to the classical case, these representation are no longer isomorphic to the symmetric powers of some representation of $Sl(n|n)$. Furthermore, for fixed n, also their dimensions do not depend on k in a polynomial way any more. Nevertheless, they almost do, as we will show. For this let

$$P(n,m) = \dim(S^{m-1})$$

denote the dimension of the irreducible representation S^{m-1} of $Sl(n|n)$ for integers $m \geq 1$, using the index shift in analogy with the dimensions $P_{cl}(n,m)$ in the classical case. Formally put $P(0,m) = P(m,0) = 0$. We show that $P(n,m) = P(m,n)$ holds whenever both sides are defined. Secondly, we show the existence of certain polynomials $p(n,x)$ of degree $2(n-1)$ in the variable x such that the following holds

$$P(n,m) = p(n,m) \quad , \qquad \text{for all} \ \ m \geq n \ .$$

But for $0 \leq m < n$ we have

$$P(n,m) = p(n,m) + (-1)^{m+n-1}(m-n) \ .$$

In particular, this shows that $P(n,m)$ is not a polynomial in m. In the following we will provide an easy algorithm to compute the polynomials $p(n,x)$. See proposition 2.1. Although the values $p(n,m)$ are related to dimensions of certain $Sl(n|n)$-modules, their combinatorial properties to our surprise did not show up in the literature before, nor so did the polynomials $p(n,x)$. For the first values $p(n,m)$ and $P(n,m)$ see table 1 and table 2.

Characterization of $p(n,m)$. The above formulas uniquely define $p(n,m)$ as a numerical function on $p : \mathbb{N}_0^2 \to \mathbb{Z}$. By imposing certain axioms, one can uniquely characterize p as in proposition 2.1. Independent from that, an important step to transfer the representation theoretic description of $p(n,m)$ into a purely combinatorial description, is obtained by the following characterization of the function p in terms of the summation conditions

$$p(n,m+1)+2\cdot p(n,m)+p(n,m-1) \;=\; A(n,m)^2 \tag{1}$$

for all $n\in\mathbb{N}_0$ and all $m\in\mathbb{N}$. Here $A:\mathbb{N}_0^2\to\mathbb{N}_0$ is defined by

$$A(n,m) \;=\; \sum_{\nu=0}^{\min(m,n)} \binom{n}{\nu}\binom{n-1+m-\nu}{n-1} .$$

It is not hard to see that for fixed n there exist polynomials $a_2(n,x)$ of degree $2(n-1)$ such that $A(n,m)=a_2(n,x)|_{x=m}$ holds for all integers $m\geq 1$. Since the summation operator S^2 defined by $S^2 f(x) \;=\; f(x+1)+2\cdot f(x)+f(x-1)$ is bijective on the polynomial ring $\mathbb{Q}[x]$, this allows to characterize the polynomials $f(x)=p(n,x)$ as the unique solutions of the polynomial equations

$$S^2 f(x) = a_2(n,x)^2 \ .$$

The strategy of proof. In proposition 2.1 we iteratively define a family of polynomials $p(n,x)$ axiomatically and determine an explicit formula for these in proposition 2.3. In theorem 3.1 we also define polynomials $a_1(m,x)$ such that $a_1(m,x)|_{x=n}=A(m,n)$. We show the first variable summation equations

$$p(n+1,x)+2p(n,x)+p(n-1,x) \;=\; a_1(n,x)^2$$

and the second variable summation equations

$$p(n,x+1)+2p(n,x)+p(n,x-1) \;=\; a_2(n,x)^2 \ .$$

As polynomials identities these equations hold for all x. Notice the polynomial identity $na_1(n,x)=xa_2(n,x)$ (proposition 3.4), which amounts to the symmetry

$$nA(m,n) \;=\; mA(n,m)\ .$$

So $p(m,n)$ satisfies summation equations in the first and in the second variable. We also consider mixed summation equations showing that

$$p(n,x)+p(n,x-1)+p(n-1,x)+p(n-1,x-1)=2\cdot d(n,x)^2 \tag{2}$$

holds for the Delannoy polynomials $d(n,x)$ as defined in section 4.

2. A family of polynomials

Proposition 2.1. *For $n=0,1,2,\ldots$ there is a unique family of polynomials $p(n,x)$ in $\mathbb{Q}[x]$ with the following properties.*

(i) $p(0,x)=0$.
(ii) *Degree condition: The degree is* $\deg_x p(n,x)\leq 2(n-1)$ *for all* $n\geq 1$.
(iii) *Parity:* $p(n,x)=p(n,-x)$ *holds for all* $n\in\mathbb{N}_0$.
(iv) *Symmetry: The function* $f(n,m)=p(n,m)+(-1)^{m+n}\cdot n$ *is a symmetric function on* $\mathbb{N}_0\times\mathbb{N}_0$*, i.e.* $f(m,n)=f(n,m)$ *holds.*

Proof of proposition 2.1. We show that the properties (i)–(iv) uniquely define the polynomials $p(n,x)$ by recursion. For $n=0$ the polynomial $p(0,x)=0$ is fixed by property (i). For $n=1$, by (ii) we know $p(1,x)=c$ is a constant polynomial. By (iv)

$$p(1,0)+(-1)^{1+0}\cdot 1 \ = \ p(0,1)+(-1)^{0+1}\cdot 0\,,$$

so $c=1$. Assuming $p(k,x)$ to be constructed for $0\leq k\leq n$ we obtain by property (iv) the following values of $p(n+1,x)$

$$p(n+1,k) \ = \ p(k,n+1)+(-1)^{n+k}(n+1-k)\,.$$

Using (iii) we find $p(n+1,-k)=p(n+1,k)$ and we thus have fixed the values $p(n+1,x)$ at the $2n+1$ places $x\in\{-n,\dots,0,\dots,n\}$. But by (ii) the degree of $p(n+1,x)$ is at most $2n$, hence $p(n+1,x)$ is the unique interpolation polynomial of degree $2n$ for the above described values. □

For example, condition (iv) together with (i) implies

$$p(n,0) \ = \ (-1)^{n-1}\cdot n \qquad \text{and} \qquad p(n,1) \ = \ 1+(-1)^n(n-1)\,.$$

In particular,

$$\begin{aligned}
&p(0,x)=0\,,\\
&p(1,x)=1\,,\\
&p(2,x)=4x^2-2\,,\\
&p(3,x)=4x^4-8x^2+3\,,\\
&p(4,x)=\frac{16}{9}x^6-\frac{56}{9}x^4+\frac{112}{9}x^2-4\,,\\
&p(5,x)=\frac{4}{9}x^8-\frac{16}{9}x^6+\frac{92}{9}x^4-\frac{152}{9}x^2+5\,,\\
&p(6,x)=\frac{16}{225}x^{10}-\frac{8}{45}x^8+\frac{848}{225}x^6-\frac{592}{45}x^4+\frac{1612}{75}x^2-6\,,\\
&p(7,x)=\frac{16}{2025}x^{12}+\frac{32}{2025}x^{10}+\frac{596}{675}x^8-\frac{7984}{2025}x^6+\frac{34696}{2025}x^4-\frac{5872}{225}x^2+7\,,\\
&p(8,x)=\frac{64}{99225}x^{14}+\frac{32}{4725}x^{12}+\frac{64}{405}x^{10}-\frac{46384}{99225}x^8+\frac{27968}{4725}x^6-\frac{41312}{2025}x^4+\frac{339392}{11025}x^2-8\,.
\end{aligned}$$

The proof of proposition 2.1 shows that for all $n\geq 0$ the values $p(n,k)$ for $k=-n,\dots,n$ are integers. Hence, by the almost symmetry (iv), for an integer $j>0$ the value

$$p(n,n+j) \ = \ p(n+j,n)+(-1)^j\cdot j$$

is integral. This proves

Corollary 2.2. *The function $p:\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{Z}$ defined by the values $p(n,m)$ on natural numbers of the family of polynomials $p(n,x)$ in proposition 2.1 is integer-valued.*

Let n be a natural number. For integers $0\leq\mu\leq n-1$ put

$$\mu^* \ = \ n-1-\mu\,.$$

TABLE 1. Values $p(n,m)$ of the polynomials $p : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$.

n \ m	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	−2	2	14	34	62	98	142	194	254
3	3	−1	35	255	899	2303	4899	9215	15875
4	−4	4	60	900	5884	24196	75324	194820	441340
5	5	−3	101	2301	24197	151805	676197	2376701	7031301
6	−6	6	138	4902	75322	676198	4160778	19475142	74307834
7	7	−5	199	9211	194823	2376699	19475143	118493179	573785095
8	−8	8	248	15880	441336	7031304	74307832	573785096	3465441272

For integers $0 \leq \nu, \mu \leq n-1$ define the polynomials

$$t(\nu,\mu,n;x) \;=\; \prod_{k=1}^{\mu}(x+\nu-\mu+k)\cdot\prod_{l=1}^{\nu}(x-1-\mu+l)\,.$$

Proposition 2.3. *The polynomials $p(0,x)=0$ and*

$$p(n,x) \;=\; \sum_{\nu,\mu=0}^{n-1}\frac{t(\nu,\mu,n;x)\cdot t(\mu^*,\nu^*,n;x)}{\nu!\nu^*!\mu!\mu^*!}$$

for $n>0$ satisfy the properties of proposition 2.1.

Proof of proposition 2.3. By definition, condition (i) of proposition 2.1 is satisfied. For the summands of $p(n,x)$ we have for all ν,μ

$$\deg_x t(\nu,\mu,n;x)\cdot t(\mu^*,\nu^*,n;x) \;=\; \nu+\mu+\nu^*+\mu^* \;=\; 2(n-1)\,,$$

so the same holds true for $p(n,x)$. This implies property (ii). Obviously

$$t(\nu,\mu,n;-x) \;=\; (-1)^{\nu+\mu}\cdot t(\mu,\nu,n;x)\,,$$

so condition (iii) follows

$$p(n,-x) \;=\; \sum_{\mu,\nu=0}^{n-1}(-1)^{2(n-1)}\frac{t(\mu,\nu,n;x)\cdot t(\nu^*,\mu^*,n;x)}{\nu!\nu^*!\mu!\mu^*!} \;=\; p(n,x)\,.$$

In order to prove (iv), which is trivial for $n=m$, we assume $m>n$. Substituting $\mu\mapsto n-1-\mu$ we write

$$p(n,m) \;=\; \sum_{\nu^*,\mu^*=0}^{m-1}\frac{t(\nu,\mu^*,n;m)\cdot t(\mu,\nu^*,n;m)}{\nu!\nu^*!\mu!\mu^*!}\,.$$

Notice that for $m\leq\mu^*$ the value

$$t(\nu,\mu^*,n;m) \;=\; (m+\nu)\cdots(m+\nu-\mu^*+1)\cdot(m+\nu-\mu^*-1)\cdots(m-\mu^*)$$

is zero unless $m+\nu-\mu^*=0$, where the value is $(-1)^\nu \nu!\mu^*!$. We obtain

$$p(n,m) \;=\; \sum_{\nu^*,\mu^*=0}^{m-1} \frac{t(\nu,\mu^*,n;m)\cdot t(\mu,\nu^*,n;m)}{\nu!\nu^*!\mu!\mu^*!} \;+\; \sum_{\nu+\mu=n-1-m} (-1)^{\nu+\mu}\,.$$

Substituting $i=n-1-\nu$ and $j=n-1-\mu$ the first sum becomes

$$\sum_{i,j=0}^{m-1} \frac{t(i,m-1-j,m;n)t(j,m-1-i,m;n)}{i!(m-1-i)!j!(m-1-j)!} \;=\; p(m,n)\,.$$

So for $m>n$

$$p(n,m) \;=\; p(m,n)+(-1)^{m+n-1}(n-m)\,.$$

Hence condition (iv) of proposition 2.1 holds for all integers $m,n>0$. □

Definition 2.4. For integers α and β we define the natural numbers

$$D_m(\alpha+1,\beta) \;=\; \begin{cases} \frac{m}{\alpha+\beta+1}\binom{m+\alpha}{\alpha}\binom{m-1}{\beta} & \text{if } \alpha\geq 0 \text{ and } 0\leq\beta\leq m-1 \\ 0 & \text{else}\end{cases}\,.$$

Remark 2.5. i) Property (ii) of proposition 2.1 can be sharpened to become

(ii') $\deg_x p(n,x)=2(n-1)$ for all $n>0$.

ii) Fixing the first variable, the function $p(n,x)$ is polynomial in x by definition. By property (iv) of proposition 2.1 the values $p(n,m)$ are nearly symmetric

$$p(m,n) \;=\; p(n,m)+(-1)^{m+n}(n-m)\,.$$

Hence for fixed $n\in\mathbb{N}$ the function $p(n,m)$ is almost a polynomial of degree $2(m-1)$ in the first variable n.

iii) For integers $m>0$ there is a convenient presentation of $t(\nu,\mu,n;m)$

$$\frac{t(\nu,\mu,n;m)}{\nu!\mu!} \;=\; \begin{cases} \frac{m}{m+\nu-\mu}\binom{m+\nu}{\nu}\binom{m-1}{\mu} & \text{if } \mu\neq m+\nu \\ (-1)^\nu & \text{if } \mu=m+\nu\end{cases}\,.$$

In particular, for integers $m\geq n>0$ we obtain

$$p(n,m) \;=\; \sum_{\nu,\mu=0}^{n-1} \frac{m^2}{(m+\nu-\mu)^2}\binom{m+\nu}{\nu}\binom{m-1}{\mu}\binom{m+(n-1-\mu)}{n-1-\mu}\binom{m-1}{n-1-\nu}\,.$$

Equivalently, using definition 2.4, for integers $m\geq n>0$

$$p(n,m) \;=\; \sum_{\nu,\mu=0}^{n-1} D_m\big(\nu+1,m-1-\mu\big)\cdot D_m\big((n-1-\mu)+1,m-1-(n-1-\nu)\big)\,.$$

Hence the values $p(n,m)$ are natural numbers for all integers $m\geq n$. In general, for integers $m,n>0$ let us define the numbers

$$P(n,m) \;=\; \sum_{\nu,\mu=0}^{\min\{n-1,m-1\}} D_m\big(\nu+1,m-1-\mu\big)\cdot D_m\big((n-1-\mu)+1,m-1-(n-1-\nu)\big)\,.$$

Hence $p(n,m)=P(n,m)$ holds for $m\geq n>0$, whereas for $m<n$ we obtain

$$p(n,m) \;=\; P(n,m)+(-1)^{m+n-1}(n-m)\,.$$

Table 2. Values $P(n,m)$ of the function $P : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$.

n \ m	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	1	14	34	62	98	142	194	254
3	0	1	34	255	899	2303	4899	9215	15875
4	0	1	62	899	5884	24196	75324	194820	441340
5	0	1	98	2303	24196	151805	676197	2376701	7031301
6	0	1	142	4899	75324	676197	4160778	19475142	74307834
7	0	1	194	9215	194820	2376701	19475142	118493179	573785095
8	0	1	254	15875	441340	7031301	74307834	573785095	3465441272

On the other hand, we know $p(n,m) = p(m,n) + (-1)^{m+n-1}(n-m)$ by property (iv). For $m < n$ we obtain

$$P(n,m) \;=\; p(m,n) \;=\; P(m,n)\,.$$

Hence the numbers $P(n,m)$ are symmetric.

3. Summation operators

Consider the summation operator S acting on functions f by

$$Sf(x) \;=\; f(x+\frac{1}{2}) + f(x-\frac{1}{2})\,.$$

On monomials S acts by $S(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} 2^{-k}(1+(-1)^k)$. The preimages $S^{-1}(x^n)$ are determined uniquely by recursion starting with $S^{-1}(0) = 0$ and $S^{-1}(1) = \frac{1}{2}$. This shows that S is bijective on polynomial rings over fields of characteristic $\neq 2$. Furthermore, the summation operator S^2 is well-defined also on functions $f : \mathbb{Z} \to \mathbb{C}$

$$S^2 f(x) \;=\; f(x+1) + 2\cdot f(x) + f(x-1)\,.$$

For integers k consider the following polynomials of degree k with $\begin{bmatrix} x \\ 0 \end{bmatrix} = 1$ and

$$\begin{bmatrix} x \\ k \end{bmatrix} \;=\; \frac{x(x-1)\cdots(x-k+1)}{k!} \qquad \text{for } k \geq 1\,.$$

Their values coincide with the binomial coefficients $\binom{n}{k} = \begin{bmatrix} n \\ k \end{bmatrix}$ for all integers $n \geq 0$.

Theorem 3.1. *Let $p : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$ be the function defined in corollary 2.2. Let S_1 and S_2 be the summation operators in the first and second variable, respectively, and consider the function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by*

$$S_1^2 p(n,m) \;=\; p(n+1,m) + 2\cdot p(n,m) + p(n-1,m) \;=\; A(m,n)^2\,. \tag{3}$$

Then we also have

$$S_2^2 p(n,m) \;=\; p(n,m+1) + 2\cdot p(n,m) + p(n,m-1) \;=\; A(n,m)^2 \tag{4}$$

TABLE 3. Values $A(n,m)$ of the function $A : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$.

$n \backslash m$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	2	2	2	2	2	2	2	2
2	1	4	8	12	16	20	24	28	32
3	1	6	18	38	66	102	146	198	258
4	1	8	32	88	192	360	608	952	1408
5	1	10	50	170	450	1002	1970	3530	5890
6	1	12	72	292	912	2364	5336	10836	20256
7	1	14	98	462	1666	4942	12642	28814	59906
8	1	16	128	688	2816	9424	27008	68464	157184

and $A(n,m)$ defines the numbers [9, A266213] *resp. the transposed array of the numbers* [9, A122542]. *They satisfy the tribonacci identities*[1]

$$A(n,m) = A(n-1,m) + A(n,m-1) + A(n-1,m-1) \ .$$

For all integers $n = 1, 2, \ldots$ we define polynomials of degree n

$$a_1(n,x) \ = \ \sum_{\nu=0}^{n} \begin{bmatrix} x-1+\nu \\ \nu \end{bmatrix} \begin{bmatrix} x \\ n-\nu \end{bmatrix}$$

and for all integers $m = 1, 2, \ldots$ we define polynomials of degree $m-1$

$$a_2(m,x) \ = \ \sum_{\nu=0}^{m} \binom{m}{\nu} \begin{bmatrix} x-\nu+m-1 \\ m-1 \end{bmatrix} \ .$$

Then for all integers $n, m > 0$ the following holds

$$a_1(m,n) \ = \ A(n,m) \quad , \quad a_2(n,m) \ = \ A(n,m) \ .$$

Let $p(n,x)$ be the family of polynomials of proposition 2.1 defining the numbers $p(n,m)$. Then for all integers $n > 0$

$$S_1^2 p(n,x) \ = \ p(n+1,x) + 2 \cdot p(n,x) + p(n-1,x) \ = \ a_1(n,x)^2 \tag{5}$$

holds. Furthermore, for all integers $n > 0$ we have

$$S_2^2 p(m,x) \ = \ p(m,x+1) + 2 \cdot p(m,x) + p(m,x-1) \ = \ a_2(m,x)^2 \ . \tag{6}$$

Before we prove theorem 3.1 let us make the following remarks: Consider the symmetric numbers $P(n,m)$ defined in remark 2.5(iii). For $m > n$ they satisfy $P(n,m+i) = p(n,m+i)$ for $i = -1, 0, 1$. Similarly for $m < n$, they satisfy $P(n,m+i) = p(n,m+i) + (-1)^{m+i+n}(n-m-i)$ for $i = -1, 0, 1$ by Proposition 2.1 (iv). For $m = n$ we obtain $P(n,n+i) = p(n,n+i)$ for $i = 0, 1$, but $P(n,n+1) = p(n,n+1) - 1$. By theorem 3.1, this implies the following corollary.

[1] Indeed $A(n,m) = D(n,m) - D(n,m-1)$ holds for the Delannoy numbers $D(n,m)$ (see our remarks on page 15). Since the $D(n,m)$ satisfy the tribonacci identities [1], hence also the $A(n,m)$ do.

Corollary 3.2. *For all integers $m,n>0$ the symmetric numbers*

$$P(n,m) \;=\; \sum_{\nu,\mu=0}^{\min(n-1,m-1)} \frac{n^2}{(m+\nu-\mu)^2}\binom{m+\nu}{\nu}\binom{m-1}{\mu}\binom{n+m-1-\mu}{n-1-\mu}\binom{m-1}{n-1-\nu}$$

satisfy the summation equation for all $m\neq n$

$$P(n,m+1)+2\cdot P(n,m)+P(n,m-1) \;=\; A(n,m)^2\;,$$

whereas for $m=n$

$$P(n,n+1)+2\cdot P(n,n)+P(n,n-1)+1 \;=\; A(n,n)^2\;.$$

Theorem 3.3. *Let S^{m-1} be the irreducible $Sl(n|n)$ Lie supermodule of maximal atypical weight $(m-1,0,\ldots,0\,|\,0,\ldots,0,1-m)$. Its dimension is given by $P(n,m)$ as in corollary 3.2*

$$\dim(S^{m-1}) \;=\; P(n,m)\;.$$

Proof of theorem 3.3. We use the $Sl(n|n)$-module

$$\mathbb{A}_{S^m} \;=\; \mathrm{Sym}^m(k^{n|n})\otimes\Lambda^m(k^{n|n})^\vee\;,$$

whose dimension is the product of the dimensions of its tensor factors. This immediately implies $\dim(\mathbb{A}_{S^m})=A(n,m)^2$. By [4, lemma 4.1] for all $m\geq 1$ we have the following decomposition in the Grothendieck ring of representations

$$\mathbb{A}_{S^m} \;\sim S^{m-2}+S^m+2\cdot S^{m-1}\;, \tag{7}$$

for $m\neq n$, and for $m=n$ we have

$$\mathbb{A}_{S^n} \;\sim S^{n-2}+S^n+2\cdot S^{n-1}+1\;. \tag{8}$$

Here formally put $S^{-1}=0$. We obtain summation equations for the dimensions

$$\dim(S^m)+2\cdot\dim(S^{m-1})+\dim(S^{m-2}) \;=\; A(n,m)^2\;,$$

for $m\neq n$, respectively for $m=n$

$$\dim(S^n)+2\cdot\dim(S^{n-1})+\dim(S^{n-2})+1 \;=\; A(n,n)^2\;.$$

By corollary 3.2 the numbers $P(n,m)$ for fixed n are also solutions of these equations. Any solution of these recursion equations, in particular $\dim(S^{m-1})$, is then of the form $P(n,m)+(-1)^m(c_1(n)m+c_0(n))$ for certain constants c_1 and c_0. For $\dim(S^{m-1})$ these constants are $c_1=c_0=0$, due to the initial conditions $\dim(S^{-1})=0=P(n,0)$ and $\dim(S^0)=1=P(n,1)$. □

We now list some of the polynomials $a_1(n,x)$ and $a_2(m,x)$:

$$
\begin{aligned}
a_1(1,x) &= 2x\,, & a_2(1,x) &= 2\,,\\
a_1(2,x) &= 2x^2\,, & a_2(2,x) &= 4x\,,\\
a_1(3,x) &= \frac{4}{3}x(x^2+\frac{1}{2})\,, & a_2(3,x) &= 4(x^2+\frac{1}{2})\,,\\
a_1(4,x) &= \frac{2}{3}x^2(x^2+2)\,, & a_2(4,x) &= \frac{8}{3}x(x^2+2)\,,\\
a_1(5,x) &= \frac{4}{15}x(x^4+5x^2+\frac{3}{2})\,, & a_2(5,x) &= \frac{4}{3}(x^4+5x^2+\frac{3}{2})\,,\\
a_1(6,x) &= \frac{4}{45}x^2\big(x^4+10x^2+\frac{23}{2}\big)\,, & a_2(6,x) &= \frac{8}{15}x\big(x^4+10x^2+\frac{23}{2}\big)\,,\\
a_1(7,x) &= \frac{8}{315}x\big(x^6+\frac{35}{2}x^4+49x^2+\frac{45}{4}\big)\,, & a_2(7,x) &= \frac{8}{45}\big(x^6+\frac{35}{2}x^4+49x^2+\frac{45}{4}\big)\,,\\
a_1(8,x) &= \frac{2}{315}x^2(x^2+6)\big(x^4+22x^2+22\big)\,, & a_2(8,x) &= \frac{16}{315}x(x^2+6)\big(x^4+22x^2+22\big)\,.
\end{aligned}
$$

The polynomials $a_1(n,x)$ and $a_2(m,x)$ satisfy the following properties.

Proposition 3.4. (a) *For all $n>0$ there is an identity of polynomials*

$$n\cdot a_1(n,x) \;=\; x\cdot a_2(n,x)\,.$$

In particular, $nA(m,n)=mA(n,m)$ is a symmetric function on $\mathbb{N}^2$.

(b) *The polynomial*

$$a_2(m,x) \;=\; \frac{2^m}{(m-1)!}\cdot x^{m-1}+\cdots+\big(1+(-1)^{m-1}\big)$$

of degree $(m-1)$ is even or odd: $a_2(m,x)=(-1)^{m-1}a_2(m,-x)$. Its value at $x=1$ is $a_2(m,1)=2m$.

Proof of proposition 3.4. The ν-th summand of the sum in $a_1(n,-x)$ is

$$\frac{1}{\nu!(n-\nu)!}(-x-1+\nu)\cdots(-x+1)(-x)\cdot(-x)(-x-1)\cdots(-x-n+\nu+1)\,,$$

and this equals

$$\frac{(-1)^n\cdot x}{n}\cdot\binom{n}{\nu}\begin{bmatrix}x-\nu+n-1\\ n-1\end{bmatrix}\,,$$

which up to the factor $\frac{(-1)^n x}{n}$ is the ν-th summand of $a_2(n,x)$. This implies $(-1)^n n\cdot a_1(n,-x) \;=\; x\cdot a_2(n,x)$. Property (iii) of proposition 2.1, i.e. $p(n,x)=p(n,-x)$, is inherited to the images $S_1^2p(n,x)=a_1(n,x)^2$ under the summation operator S_1^2. So $a_1(n,x)$ is either even or odd, which depends on whether its degree is even or odd. Part (a) follows. Accordingly, $a_2(n,-x)=(-1)^{n-1}a_2(n,x)$ is even or odd. The leading term of the polynomial $\begin{bmatrix}x\\ m-1\end{bmatrix}$ is $\frac{1}{(m-1)!}x^{n-1}$. So the leading term of $a_2(m,x)$ is

$$\left(\sum_{\nu=0}^{m}\binom{m}{\nu}\right)\cdot\frac{x^{m-1}}{(m-1)!}\;=\;\frac{2^m}{(m-1)!}\cdot x^{m-1}\,.$$

Evaluating $a_2(m,x)$ at $x=0$ only has constituents from $\nu=0$ and m, so we obtain $a_2(m,0) = 1+(-1)^{m-1}$. Similarly, at $x=1$ only the terms for $\nu=0$ and 1 are non-zero, and we obtain $a_2(m,1)=2m$. □

This being said, we come to the

Proof of theorem 3.1. We show (5) at the integer points $x=m>n$. Then, as both $S_1^2p(n,x)$ and $a_1(n,x)^2$ are polynomials, they must be equal. Let $m>n$ be an integer. By changing the summation index we have

$$a_1(n,m)^2 = \left(\sum_{\nu=0}^{n}\binom{m-1+\nu}{n-1}\binom{m}{n-\nu}\right)\cdot\left(\sum_{\mu=0}^{n}\binom{m-1+n-\mu}{m-1}\binom{m}{\mu}\right).$$

Hence $a_1(n,m)^2 = \sum_{\nu,\mu=0}^{n} a(\nu,\mu,n,m)$ for

$$a(\nu,\mu,n,m) = \binom{m-1+\nu}{\nu}\binom{m}{n-\nu}\binom{m-1+n-\mu}{n-\mu}\binom{m}{\mu}. \tag{9}$$

On the other hand, using the notation of remark 2.5 (iii),

$$\begin{aligned} S_1^2p(n,m) &= \sum_{\nu,\mu=0}^{n} D_m(\nu+1,m-1-\mu)\cdot D_m(n-\mu+1,m-1-n+\nu)\\ &+2\cdot\sum_{\nu,\mu=0}^{n-1} D_m(\nu+1,n-1-\mu)\cdot D_m(n-1-\mu+1,m-1-(n-1)+\nu)\\ &+\sum_{\nu,\mu=0}^{n-2} D_m(\nu+1,m-1-\mu)\cdot D_m(n-2-\mu+1,m-1-(n-2)+\nu).\end{aligned}$$

Index shifts $\nu\mapsto\nu+1$, $\mu\mapsto\mu+1$ in the last sum and in one of the second sums yield

$$\begin{aligned} S_1^2p(n,m) &= \sum_{\nu,\mu=0}^{n} D_m(\nu+1,m-1-\mu)\cdot D_m(n-\mu+1,m-1-n+\nu)\\ &+\sum_{\nu,\mu=0}^{n-1} D_m(\nu+1,m-1-\mu)\cdot D_m(n-\mu,m-n+\nu)\\ &+\sum_{\nu,\mu=1}^{n} D_m(\nu,m-\mu)\cdot D_m(n-\mu+1,m-1-n+\nu)\\ &+\sum_{\nu,\mu=1}^{n-1} D_m(\nu,m-\mu)\cdot D_m(n-1-\mu+1,m-n+\nu).\end{aligned}$$

Using this, equation (5) follows now from the next lemma 3.5. The identity $a_1(n,m) = a_2(m,n)$ for integers $m,n>0$ becomes obvious from substituting $\nu\mapsto n-\nu$ in $a_1(n,m)$, noticing that in both sums the summands are actually zero for $\nu>\min(m,n)$. Since $a_2(n,m)=A(n,m)$ by the definition of A given in the introduction, formula (3) follows from (5). Then also equation (6) holds for all integers $x=n>0$, and then for all $m>0$

by the nearby symmetry of $p(n,m)$. Exchanging m and n we obtain (4). Therefore, as polynomials of degree $2(m-1)$ in x, $a_2(m,x)^2$ and $S_2^2 p(m,x)$ coincide. $\square$

Lemma 3.5. *For $m > n$ the numbers $a(\nu,\mu,n,m)$ defined in formula (9) satisfy*

$$a(\nu,\mu,n,m) \;=\; b(\nu,\mu,n,m)\ ,$$

where $b(\nu,\mu,n,m)$ abbreviates the following sum of four terms

$$\begin{aligned}
&D_m(\nu+1,m-1-\mu)\cdot D_m(n-\mu+1,m-1-n+\nu)\\
&\ +(1-\delta_{\nu,n})\cdot(1-\delta_{\mu,n})\cdot D_m(\nu+1,m-1-\mu)\cdot D_m(n-\mu,m-n+\nu)\\
&\ +(1-\delta_{\nu,0})\cdot(1-\delta_{\mu,0})\cdot D_m(\nu,m-\mu)\cdot D_m(n-\mu+1,m-1-n+\nu)\\
&\ +(1-\delta_{\nu,0})\cdot(1-\delta_{\mu,0})(1-\delta_{\nu,n})\cdot(1-\delta_{\mu,n})\cdot D_m(\nu,m-\mu)\cdot D_m(n-\mu,m-n+\nu)\ .
\end{aligned}$$

Proof of lemma 3.5. Using the definition of $D_m(\alpha+1,\beta)$ (see 2.4), the lemma follows by straight forward calculations distinguishing the cases ν,μ equal to 0, n, or generic. We exemplify this in the generic case $0<\nu,\mu<n$. The sum $b(\nu,\mu,n,m)$ here is

$$\begin{aligned}
&\frac{m^2}{(m+\nu-\mu)^2}\binom{m+\nu}{\nu}\binom{m-1}{m-1-\mu}\binom{m+n-\mu}{n-\mu}\binom{m-1}{m-1-n+\nu}\\
&+\frac{m^2}{(m+\nu-\mu)^2}\binom{m+\nu}{\nu}\binom{m-1}{m-1-\mu}\binom{m+n-1-\mu}{n-1-\mu}\binom{m-1}{m-n+\nu}\\
&+\frac{m^2}{(m+\nu-\mu)^2}\binom{m+\nu-1}{\nu-1}\binom{m-1}{m-\mu}\binom{m+n-\mu}{n-\mu}\binom{m-1}{m-1-n+\nu}\\
&+\frac{m^2}{(m+\nu-\mu)^2}\binom{m+\nu-1}{\nu-1}\binom{m-1}{m-\mu}\binom{m+n-1-\mu}{n-1-\mu}\binom{m-1}{m-n+\nu}\ .
\end{aligned}$$

Summing the first and the second line as well as the third and forth we obtain

$$\frac{m}{(m+\nu-\mu)}\left[\binom{m+\nu}{\nu}\binom{m-1}{\mu}+\binom{m+\nu-1}{\nu-1}\binom{m-1}{m-\mu}\right]\binom{m}{n-\nu}\binom{m+n-1-\mu}{n-\mu},$$

which simplifies to $a(\nu,\mu,n,m)$. $\square$

4. Mixed summation operators

Proposition 4.1. *For $n=1,2,3,\ldots$ there is a unique family of polynomials $d(n,x)$ in $\mathbb{Q}[x]$ satisfying the following properties:*

(i) *For all n the degree of the polynomial $d(n,x)$ is $n-1$.*
(ii) *For all n the leading coefficient of $d(n,x)$ is $\frac{2^{n-1}}{(n-1)!}$.*
(iii) *The function $d(n,m)$ is symmetric on $\mathbb{N}\times\mathbb{N}$, i.e. $d(n,m)=d(m,n)$.*

Proof. The conditions imply $d(1,x)=1$. By recursion, let $d(m,x)$ be determined for all $1\le m<n$. By (i) the polynomial $d(n,x)$ has degree $n-1$ and fixed leading coefficient by (ii). By (iii) the identities $d(n,m)=d(m,n)$ determine $n-1$ equations for the $n-1$ missing coefficients of $d(n,x)$, so determine $d(n,x)$ uniquely. $\square$

We formally put $d(0,x)=0$ and obtain

$$
\begin{aligned}
d(1,x) &= 1\ ,\\
d(2,x) &= 2(x-\frac{1}{2})\ ,\\
d(3,x) &= 2\big((x-\frac{1}{2})^2+\frac{1}{4}\big)\ ,\\
d(4,x) &= \frac{4}{3}(x-\frac{1}{2})\big((x-\frac{1}{2})^2+\frac{5}{4}\big)\ ,\\
d(5,x) &= \frac{2}{3}\big((x-\frac{1}{2})^4+\frac{7}{2}(x-\frac{1}{2})^2+\frac{9}{16}\big)\ ,\\
d(6,x) &= \frac{4}{15}(x-\frac{1}{2})\big((x-\frac{1}{2})^4+\frac{15}{2}(x-\frac{1}{2})^2+\frac{89}{16}\big)\ ,\\
d(7,x) &= \frac{4}{45}\big((x-\frac{1}{2})^2+\frac{9}{4}\big)\big((x-\frac{1}{2})^4+\frac{23}{2}(x-\frac{1}{2})^2+\frac{25}{16}\big)\ ,\\
d(8,x) &= \frac{8}{315}(x-\frac{1}{2})\big((x-\frac{1}{2})^6+\frac{91}{4}(x-\frac{1}{2})^4+\frac{1519}{16}(x-\frac{1}{2})^2+\frac{3429}{64}\big)\ .
\end{aligned}
$$

Proposition 4.2. *The polynomials $d(n,x)$ defined by proposition 4.1 for $n\geq 1$ have the following properties:*

(a) *Tribonacci identity: $d(n,x) \ = \ d(n,x-1)+d(n-1,x-1)+d(n-1,x)$.*
(b) *$d(n,x)$ for $x=m$ is the Delannoy number $D(n-1,m-1)$; see* [9, A008288].
(c) *$d(n,1-x) \ = \ (-1)^{n-1}d(n,x)$ holds for all n.*

Proof of proposition 4.2. For (a), the family of polynomials $v(0,x)=0$ and for $n\geq 1$

$$v(n,x) \ = \ d(n,x-1)+d(n-1,x-1)+d(n-1,x)$$

obviously satisfy the properties of proposition 4.1. Hence $v(n,x)=d(n,x)$. For (b) notice, that the Delannoy numbers $D(n-1,m-1)$ are determined by the tribonacci identity and the conditions $D(n-1,0)=D(0,m-1)=1$, in agreement with $d(1,x)=1$ and the symmetry property prop. 4.1 (iii). For (c) notice that, by definition,

$$b(n,x) := \ (-1)^{n-1}d(n,1-x)-d(n,x)$$

satisfies $b(n,1-x)=(-1)^{n-1}b(n,x)$. To show $b(n,x)=0$ and hence (c), it suffices to show

$$b(n,-x) \ = \ (-1)^n\cdot b(n,x) \tag{10}$$

since then $b(n,1+x)=-b(n,x)$, and hence $b(n,x)=0$ by considering the leading term. Equation (10) is evident for $n=0$. For $n\geq 1$ replace x by $1-x$ in (a) to obtain

$$d(n,1-x)-d(n,-x) \ = \ d(n-1,1-x)+d(n-1,-x)\ .$$

Arguing by induction on n, on the right we obtain $(-1)^n d(n-1,x)+(-1)^n d(n-1,x+1)$. Using part (a) for x replaced by $x+1$, we find that $(-1)^n d(n-1,x)+(-1)^n d(n-1,x+1)$ is equal to $(-1)^n d(n,x+1)-(-1)^n d(n,x)$. Altogether this implies

$$d(n,1-x)-d(n,-x) \ = \ (-1)^n d(n,x+1)-(-1)^n d(n,x)\ ,$$

which is equivalent to the desired equation (10). ∎

TABLE 4. Delannoy array $D(n-1, m-1)$ of the function $d : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$.

$n \backslash m$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1	1	1	1	1	1
2	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11
3	61	41	25	13	5	1	1	5	13	25	41	61
4	-231	-129	-63	-25	-7	-1	1	7	25	63	129	231
5	681	321	129	41	9	1	1	9	41	129	321	681
6	-1683	-681	-231	-61	-11	-1	1	11	61	231	681	1683
7	3653	1289	377	85	13	1	1	13	85	377	1289	3653
8	-7183	-2241	-575	-113	-15	-1	1	15	113	575	2241	7183

The proof of proposition 4.2(c) essentially amounts to extend the Delannoy array from $\mathbb{N} \times \mathbb{N}$ to non-positive m by a reflection along the line $x = \frac{1}{2}$, thereby introducing signs so that the tribonnaci rule of proposition 4.2(a) remains valid.

Proposition 4.3. *For all $n = 1, 2, 3, \ldots$ the polynomials $d(n, x)$ of Proposition 4.1 and the polynomials $p(n, x)$ of Proposition 2.1 satisfy the polynomial identities*

$$p(n,x) + p(n,x-1) + p(n-1,x) + p(n-1,x-1) \;=\; 2 \cdot d(n,x)^2 \,.$$

Proof of proposition 4.3. It is easy to see that for $n = 1, 2, 3, \ldots$ there is a unique family of polynomials $q(n, x)$ in $\mathbb{Q}[x]$ satisfying the following properties.

(i) For all n the degree of the polynomial is $\deg_x q(n, x) = 2(n - 1)$.
(ii) For all n the leading coefficient of $q(n, x)$ is $\frac{2^{2n-1}}{(n-1)!^2}$.
(iii) $q(n, x) = q(n, 1 - x)$ holds for all $n \in \mathbb{N}$.
(iv) The function $q(n, m)$ is symmetric on $\mathbb{N} \times \mathbb{N}$, i.e. $q(n, m) = q(m, n)$.

Obviously, the family $2d(n, x)^2$ satisfies these axioms. Hence $q(n, x) = 2d(n, x)^2$.

The polynomials $p(n, x)$ satisfy the properties of proposition 2.1. By remark 2.5(i) we know $\deg_x p(n, x) = 2(n-1)$. Hence $p(n, x) + p(n, x-1) + p(n-1, x) + p(n-1, x-1)$ satisfies the preceding properties (i), (iii), and (iv). By proposition 2.3 the leading coefficient of $p(n, x)$ can be easily determined as

$$\sum_{\nu,\mu=0}^{n-1} \frac{1}{\nu!(n-1-\nu)!\mu!(n-1-\mu)!} \;=\; \left(\frac{2^{n-1}}{(n-1)!}\right)^2 .$$

Therefore the leading coefficient of $p(n, x) + p(n, x-1) + p(n-1, x) + p(n-1, x-1)$ is twice this number, and therefore equal to $\frac{2^{2n-1}}{(n-1)!^2}$. So we have found a second solution of $q(n, x)$. The proposition follows. □

Remarks on the Delannoy numbers $D(n, m)$. It is known that

$$D(n,m) = \sum_{k=0}^{\min(n,m)} 2^k \binom{n}{k}\binom{m}{k} = \sum_{k=0}^{\min(n,m)} \binom{n}{k}\binom{m+n-k}{n}$$

with generating series $\sum_{n,m} D(n,m)x^n y^m = \frac{1}{1-x-y-xy}$. See e.g. [9], [10]. This in particular implies

$$d(n,x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \begin{bmatrix} x-k+n-2 \\ n-1 \end{bmatrix}$$

and relates the numbers $A(n,m)$ and the Delannoy numbers $D(n,m)$ by the identity

$$A(n,m) = D(n,m) - D(n,m-1) \ ,$$

or equivalently $A(n,m) = D(n-1,m) + D(n-1,m-1)$. Indeed, these relations are equivalent to

$$a_2(n-1,x) = d(n,x+1) - d(n,x) \ ,$$

and the latter follows from $\begin{bmatrix} x-k+n-1 \\ n-1 \end{bmatrix} - \begin{bmatrix} x-k+n-2 \\ n-1 \end{bmatrix} = \begin{bmatrix} x-k+n-2 \\ n-2 \end{bmatrix}$. From this we obtain the alternative presentation

$$A(n,m) \ = \ \sum_{k=1}^{\min(n,m)} 2^k \binom{n}{k} \binom{m-1}{k-1} \ .$$

For the relation of $A(n,m)$ with diamond numbers see [10].

5. Central values

Using of the summation operator $Sf(x) = f(x+\frac{1}{2}) + f(x-\frac{1}{2})$, proposition 4.3 can be reformulated

$$2 \cdot d(n, x+\frac{1}{2})^2 \ = \ S\big(p(n,x) + p(n-1,x)\big) \ . \tag{11}$$

This suggests that our families of polynomials have interesting properties at half-integral places. We illustrate this by determining their values at $x = \frac{1}{2}$.

Proposition 5.1. (a) *For the polynomials $a_1(n,x)$ and $a_2(n,x)$ of theorem 3.1 the values at $x = \frac{1}{2}$ are given by*

$$a_1(2m, \frac{1}{2}) \ = \ a_1(2m+1, \frac{1}{2}) \ = \ \begin{bmatrix} m-\frac{1}{2} \\ m \end{bmatrix} \ ,$$

respectively

$$a_2(n, \frac{1}{2}) \ = \ 2n \cdot a_1(n, \frac{1}{2}) \ .$$

(b) *For all $m \in \mathbb{N}_0$ define the rational number*

$$r(m) \ = \ \sum_{k=0}^{m} \begin{bmatrix} k-\frac{1}{2} \\ k \end{bmatrix}^2 \ = \ \sum_{k=0}^{m} \left(\frac{1}{2^{2k}} \binom{2k}{k} \right)^2 \ .$$

Then the values at $x = \frac{1}{2}$ of the polynomials $P(n,x)$ of proposition 2.1 are given by the recursion formula

$$p(2m+1, \frac{1}{2}) \ = \ r(m) \ = \ (-1) \cdot p(2m+2, \frac{1}{2}) \ . \tag{12}$$

(c) *The values at* $x = \frac{1}{2}$ *of the polynomials* $d(n,x)^2$ *are*

$$d(2m, \frac{1}{2}) \;=\; 0\,, \quad \textit{respectively} \quad d(2m+1, \frac{1}{2}) \;=\; \begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix} .$$

By the set of initial values of proposition 5.1 and the tribonacci identities satisfied by $a_2(n,x)$ and $d(n,x)$ we obtain recursion formulas for their values at $x = \frac{2k+1}{2}$. Identity (6) then gives a recursion for the values $p(n, \frac{2k+1}{2})$.

Proof of proposition 5.1. For part (a) we first notice that

$$\begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix} \;=\; \frac{1}{2^{2m}} \binom{2m}{m} \;=\; (-1)^m \begin{bmatrix} -\frac{1}{2} \\ m \end{bmatrix} .$$

So the generating series of $\begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix}$ is given by the Taylor series

$$\sum_{m=0}^{\infty} x^m \begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix} \;=\; \sum_{m=0}^{\infty} (-x)^m \begin{bmatrix} -\frac{1}{2} \\ m \end{bmatrix} \;=\; \big(1-x\big)^{-\frac{1}{2}} ,$$

whereas

$$\big(1+x\big)^{\frac{1}{2}} \;=\; \sum_{m=0}^{\infty} x^m \begin{bmatrix} \frac{1}{2} \\ n \end{bmatrix} .$$

By Cauchy product expansion we obtain the generating series for $a_1(n, \frac{1}{2})$

$$\sqrt{\frac{1+x}{1-x}} \;=\; \sum_{n=0}^{\infty} x^n \sum_{\nu=0}^{n} \begin{bmatrix} \frac{1}{2} \\ \nu \end{bmatrix} \begin{bmatrix} n - \nu - \frac{1}{2} \\ n - \nu \end{bmatrix} \;=\; \sum_{n=0}^{\infty} x^n a_1(n, \frac{1}{2}) .$$

On the other hand, from

$$\frac{1}{\sqrt{1-x^2}} \;=\; \sum_{m=0}^{\infty} x^{2m} \begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix}$$

we obtain

$$\sqrt{\frac{1+x}{1-x}} \;=\; \frac{1+x}{\sqrt{1-x^2}} \;=\; \sum_{m=0}^{\infty} \big(x^{2m} + x^{2m+1}\big) \begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix} .$$

Comparing coefficients, the first identity of part (a) is proved. For the second recall $na_1(n,x) = xa_2(n,x)$ from proposition 3.4.
For part (b), from the list of $p(n,x)$ following proposition 2.1 we obtain $p(1, \frac{1}{2}) = 1 = r(0) = -p(2, \frac{1}{2})$ and $p(3, \frac{1}{2}) = \frac{5}{4} = r(1) = p(4, \frac{1}{2})$. By induction assuming (12) holds true for all $m < M$, and by (5)

$$p(2M+1, \frac{1}{2}) + 2p(2M, \frac{1}{2}) + p(2M-1, \frac{1}{2}) \;=\; a_1(2M, \frac{1}{2})^2 ,$$

we conclude

$$p(2M+1, \frac{1}{2}) \;=\; \begin{bmatrix} M - \frac{1}{2} \\ M \end{bmatrix}^2 + 2r(M-1) - r(M-1) \;=\; r(M) .$$

Similarly we deduce $p(2M+2,\frac{1}{2}) = -r(M)$. Concerning part (c), by (11) we obtain

$$2\cdot d(n,\frac{1}{2})^2 \;=\; p(n,\frac{1}{2}) + p(n,-\frac{1}{2}) + p(n-1,\frac{1}{2}) + p(n-1,-\frac{1}{2})\,.$$

Since the polynomials $p(n,x)$ are even functions, we get

$$d(n,\frac{1}{2})^2 \;=\; p(n,\frac{1}{2}) + p(n-1,\frac{1}{2})\,.$$

By part (b), this is zero in case $n=2m$, whereas in case $n=2m+1$ we obtain

$$d(2m+1,\frac{1}{2})^2 = r(m) - r(m-1) = \begin{bmatrix} m-\frac{1}{2} \\ m \end{bmatrix}^2 \;.$$

In [3] the polynomials occur as Mellin transforms of Laguerre functions. By [3, Thm 4], their zeros lie on the line $x=\frac{1}{2}$. In particular, interpolating the Delannoy numbers, $d(2m+1,x)$ is positive, which implies part (c). Notice that $d(2m,\frac{1}{2}) = 0$ already follows from proposition 4.2 (c). $\square$

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