A GENERALISATION OF S. ZHANG'S LOCAL GROSS-ZAGIER FORMULA FOR GL_2

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ABSTRACT. S. Zhang's local Gross-Zagier formulae for GL_2 can be interpreted as a fundamental lemma for some relative trace formulae. From this point of view we prove the existence of the corresponding local transfer. Further we construct universally defined geometric operators which realize the behavior of Hecke operators on the analytic side. We use them to give a proof of the local Gross-Zagier formula for GL_2. We work locally and throughout computationally.

CONTENTS

1. INTRODUCTION

The Gross-Zagier formula [4] is a relation between a Heegner point of discriminant D on the moduli space $X_0(N)$ and the Rankin-Selberg L-function attached to a newform f of weight 2 and level N and a character χ of the class group of

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$K = \mathbb{Q}(\sqrt{D})$, in case D is squarefree and prime to N:

$$L'(f, \chi, s = \frac{1}{2}) \; = \; const \cdot \hat{h}(c_{\chi,f}) \; .$$

Here $\hat{h}$ is the height function on $\mathrm{Jac}(X_0(N))$ and $c_{\chi,f}$ is a component of the Heegner class depending on χ and f. S. Zhang [13], [14] switches the point of view to a local one. CM-points are studied on the corresponding Shimura curve, modular forms are automorphic representations. The height pairing of CM-cycles is replaced by a geometric pairing of Schwartz functions $\phi, \psi \in \mathcal{S}(\chi, G(\mathbb{A}_{\mathbb{F},f}))$,

$$< \phi, \psi > = \sum_\gamma m_\gamma < \phi, \psi >_\gamma \; . \tag{1}$$

Here G is an inner form of PGL_2, and T is the maximal subtorus given by the quadratic extension $\mathbb{K}/\mathbb{F}$ of the totally real field $\mathbb{F}$. The sum is over double cosets γ of $T\backslash G/T$. The multiplicities m_γ carry heavy arithmetic input. They are global data determined by intersection numbers. The coefficients $< \phi, \psi >_\gamma$ are adèlic integrals, given by their local components. Here a first parallel with a (relative) trace formula,

$$\sum_\gamma t_\gamma O_\gamma(\phi),$$

becomes visible. The sum is over double cosets γ again. The tamagawa numbers t_γ are global data, and the orbital integrals $O_\gamma(\phi)$ are computed by their local factors. The local components of the coefficients above are (for nondegenerate γ) given by

$$< \phi, \psi >_\gamma = \int_{T\backslash G} \int_T \phi(t^{-1}\gamma t y) \; dt \; \bar{\psi}(y) \; dy \; . \tag{2}$$

These expressions are close to orbital integrals on G relative to T (as in Jacquet's work on relative trace formula [6]). They can even be read as orbital integrals for (γ, id) on $G \times G$ relative to $T \times G$, the action given by $(\gamma, \delta) \cdot (t, g) = (t^{-1}\gamma t, \delta g)$ (for $t \in T, \; g, \gamma, \delta \in G$), of the function $(\gamma, \delta) \mapsto \phi(\gamma\delta)\psi(\delta)$. As this is of no further use here, we call them *local linking numbers* according to their origin [13].

S. Zhang [13] invents a kernel function for the L-function which satisfies a functional equation similar to that for L. The local Fourier coefficients of the kernel are given by products of Whittaker newforms for the theta series $\Pi(\chi)$ and the Eisenstein series Π_E occuring in the Rankin convolution. They do not depend on the cusp form anymore. (See Section 2 for concrete definitions.) We generalize these products to get invariant linear forms on the isobaric sum $\Pi(\chi) \boxplus \Pi_E$ defined by evaluating functions in the Kirillov models,

$$(W_\chi, W_E) \; \mapsto \; W_\chi(\eta) W_E(\xi) \; ,$$

for $\xi, \eta = 1 - \xi \in F \setminus \{0, 1\}$, $W_\chi \in \mathcal{K}(\Pi(\chi))$, $W_E \in \mathcal{K}(\Pi_E)$. Let $\mathcal{W}$ be the space of distributions on $\Pi(\chi) \boxplus \Pi_E$ defined by these evaluations at ξ.

Let $\phi \in \mathcal{S}(\chi, G)$ be essentially the characteristic function of the maximal compact subgroup. Then the local Gross-Zagier formula for GL_2 ([13] Lemma 4.3.1, resp. Theorem 6.4 below) essentially states that for the newforms $W_{\chi,\mathrm{new}}$, $W_{E,\mathrm{new}}$ we have an equality

$$\mathbf{T}_b\left(W_{\chi,\mathrm{new}}(\eta)W_{E,\mathrm{new}}(\xi)\right) \;=\; |b|^{-1}|\xi\eta|^{\frac{1}{2}} < \tilde{\mathbf{T}}_b\phi, \phi >_{\gamma=\gamma(\xi)} \;.$$

Here $\mathbf{T}_b$ is a Hecke operator indexed by $b \in F^\times$, and $\tilde{\mathbf{T}}_b\phi$ is a special transform of ϕ. S. Zhang et al prove local Gross-Zagier formulae with no level constraints [11] on more general Shimura curves [12].
In the language of trace formula, this is a fundamental lemma for the comparison of relative trace formulae. W. Zhang [15], [16] gives a general relative trace formula approach to the Gross-Zagier problem on unitary Shimura varieties. He formulates an arithmetic fundamental lemma in terms of unitary Rapoport-Zink spaces, proving it for small degrees. Gross-Zagier fits in the case of degree 2.

We discuss some local aspects in comparing relative trace formulae in the GL_2 case. In trace formula theory, it is a non-trivial problem to find enough local test vectors on each side at almost all places which can be compared. In the first part of the paper, we solve this problem in the case above, i.e. establish a transfer. We choose a parametrisation of the double cosets $\gamma = \gamma(\xi)$ by the projective line, $\xi \in \mathbb{P}^1(F)$, and characterize the expansion of the local linking numbers with respect to this variable (Propositions 3.1, 3.2, 3.5). This is very close to Jacquet's characterization of orbital integrals [6]. The space of distributions built by evaluating local linking numbers at ξ multiplied with the factor $|\xi\eta|^{\frac{1}{2}}$ will be denoted by $\tilde{\mathcal{L}}$. On the other hand, the expansion in ξ of the space $\mathcal{W}$ of distributions on $\Pi(\chi) \boxplus \Pi_E$ can be described by the theory of automorphic forms (Propositions 2.9, 2.10). The transfer result is:

Theorem 3.6. *The spaces $\tilde{\mathcal{L}}$ and $\mathcal{W}$ have identical ξ-expansion.*

The second part of the paper is concerned with more quantitative aspects. We construct operators on the geometric side which realize the behavior of Hecke operators on the analytic side. The existence of such operators is not surprising but to have an explicit shape of them is appealing for several aspects. It provides a tool to produce more identities like the fundamental lemma out of given ones, i.e. is a first step towards a general matching of orbital integrals. Moreover it makes the behavior around degenerate elements more visible. (Which in general forces a stabilization process.) Lastly we use these general geometric Hecke operators to rephrase S. Zhang's local Gross-Zagier formula for GL_2 and give a shorter proof.
On the analytic side, the Hecke operators $\mathbf{T}_b$ are essentially given by translations by $b \in F^\times$. In case of a split torus T they produce logarithmic and, in case of a quadratic character χ, even double logarithmic singularities as $b \to 0$,

$$\mathbf{T}_b\left(W_\chi(\eta)W_E(\xi)\right) \;=\; |b|^{-1}|\xi\eta|^{\frac{1}{2}}\chi_1(b\eta)(c_1 v(b\eta) + c_2)(c_3 v(b\xi) + c_4),$$

if $\chi_1^2 = 1$ (Proposition 5.1). The geometric Hecke operators are constructed in a simple manner to realize this pole behavior. The first natural guess is to translate the local linking numbers as well,

$$< \phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \psi >_{\gamma(\xi)} \ .$$

These translations are studied in Section 4. It turns out (Theorems 4.1, 4.3) that they do not suffice, as they do not produce the double logarithmic term $v(b)^2$. In Section 5 we construct an operator $\mathbf{S}_b$, which essentially is a weighted sum of translations by elements of valuations at most $v(b)$. This operator has good properties (Propositions 5.3, 5.4), and we get:

Theorem 5.5. *The local linking numbers $|b|^{-1}|\xi\eta|^{\frac{1}{2}}\mathbf{S}_b < \phi, \psi >_{\gamma(\xi)}$ and the Whittaker products $\mathbf{T}_b\left(W_\chi(\eta)W_E(\xi)\right)$ have the same asymptotics in b.*

Accordingly, we formulate and prove the local Gross-Zagier formula in terms of $\mathbf{S}_b$ (Theorem 6.5).
Concerning concrete calculations, the case of a compact torus T is much easier than that of a noncompact one. This is due to the inner integral of the local linking numbers having compact support in the first case. In view of the noncompact torus we have to reduce ourselves to an arbitrary but fixed ξ to describe the asymptotics in the translation variable b. Anyway the calculations for the translation (Theorem 4.3) take about one hundred pages of $\wp$-adic integration in [9]. We sketch the outline of the proof in Section 4. Due to this difficulty, the results on Hecke operators are of asymptotic nature.

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2. Terminology and preparation

2.1. **Geometry.** The geometric setting is that of S. Zhang [13]

2.1.1. *Global data.* Let $\mathbb{F}$ be a totally real algebraic number field and let $\mathbb{K}$ be a imaginary quadratic extension of $\mathbb{F}$. Further, let $\mathbb{D}$ be a division quaternion algebra over $\mathbb{F}$ which contains $\mathbb{K}$ and splits at the archimedean places. Let $\mathbf{G}$ denote the inner form of the projective group PGL_2 over $\mathbb{F}$ which is given by the multiplicative group $\mathbb{D}^\times$,

$$\mathbf{G}(\mathbb{F}) = \mathbb{F}^\times \backslash \mathbb{D}^\times .$$

Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ given by $\mathbb{K}^\times$, i.e. $\mathbf{T}(\mathbb{F}) = \mathbb{F}^\times\backslash\mathbb{K}^\times$. Let $\mathbb{A}_\mathbb{F}$ (resp. $\mathbb{A}_\mathbb{K}$) be the adèles of $\mathbb{F}$ (resp. $\mathbb{K}$) and let $\mathbb{A}_{\mathbb{F},f}$ be the subset of finite adèles. On $\mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})$ there is an action of $\mathbf{T}(\mathbb{A}_{\mathbb{F},f})$ from the left and an action of $\mathbf{G}(\mathbb{A}_{\mathbb{F},f})$ from the right. The factor space $\mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})$ can be viewed as the set of CM-points of the Shimura variety defined by the inverse system of

$$\mathrm{Sh}_K := \mathbf{G}(\mathbb{F})^+\backslash\mathcal{H}_1^n \times \mathbf{G}(\mathbb{A}_{\mathbb{F},f})/K \ ,$$

where K runs though sufficiently small compact open subgroups of $\mathbf{G}(\mathbb{A}_{\mathbb{F},f})$, $\mathcal{H}_1$ is the upper halfplane, and n is the number of the infinite places of $\mathbb{F}$. The CM-points are embedded in Sh_K by mapping the coset of $g \in \mathbf{G}(\mathbb{A}_{\mathbb{F},f})$ to the coset of (z, g), where $z \in \mathcal{H}_1^n$ is fixed by $\mathbf{T}$.
Let $\mathcal{S}(\mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f}))$ be the Schwartz space, i.e. the space of complex valued functions on $\mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})$ which are locally constant and of compact support. A character of $\mathbf{T}$ is a character χ of $\mathbf{T}(\mathbb{F})\backslash\mathbf{T}(\mathbb{A}_{\mathbb{F},f})$, that is a character of $\mathbb{A}_{\mathbb{K},f}^\times/\mathbb{K}^\times$ trivial on $\mathbb{A}_{\mathbb{F},f}^\times/\mathbb{F}^\times$. Especially, $\chi = \prod \chi_v$ is the product of its local unitary components. We have

$$\mathcal{S}(\mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})) = \oplus_\chi \mathcal{S}(\chi, \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})),$$

where $\mathcal{S}(\chi, \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f}))$ is the subspace of those functions ϕ transforming under $\mathbf{T}(\mathbb{A}_{\mathbb{F},f})$ by χ, i.e. for $t \in \mathbf{T}(\mathbb{A}_{\mathbb{F},f})$ and $g \in \mathbf{G}(\mathbb{A}_{\mathbb{F},f})$: $\phi(tg) = \chi(t)\phi(g)$. Any such summand is the product of its local components,

$$\mathcal{S}(\chi, \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})) = \otimes_v \mathcal{S}(\chi_v, \mathbf{G}(\mathbb{A}_{\mathbb{F}_v})).$$

A pairing on $\mathcal{S}(\chi, \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f}))$ can be defined as follows. For functions ϕ, ψ in $\mathcal{S}(\chi, \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f}))$ and a double coset $[\gamma] \in \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{F})/\mathbf{T}(\mathbb{F})$ define the **linking number**

$$< \phi, \psi >_\gamma := \int_{\mathbf{T}_\gamma(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})} \phi(\gamma y)\bar{\psi}(y)\ dy, \tag{3}$$

where $\mathbf{T}_\gamma = \gamma^{-1}\mathbf{T}\gamma \cap \mathbf{T}$. For γ normalizing $\mathbf{T}$ we have $\mathbf{T}_\gamma = \mathbf{T}$. Otherwise $\mathbf{T}_\gamma$ is trivial. We call γ nondegenerate, if it belongs to the latter case. Here dy denotes the quotient measure of nontrivial Haar measures on $\mathbf{G}$ and $\mathbf{T}$. Further, let

$$m\ :\ \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{F})/\mathbf{T}(\mathbb{F}) \to \mathbb{C}$$

be a multiplicity function. Then

$$< \phi, \psi > := \sum_{[\gamma]} m([\gamma]) < \phi, \psi >_\gamma$$

defines a sesquilinear pairing on $\mathcal{S}(\chi, \mathbf{T}(\mathbb{F})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f}))$. Determining the multiplicity function is an essential global problem, which was solved by S. Zhang for (local) Gross-Zagier in terms of intersection numbers. Concering the parallels with trace formula, they take over the role of Tamagawa numbers. The coefficients $< \phi, \psi >_\gamma$ are the data linking global height pairings on curves and local approaches.

2.1.2. *Local data.* In studying the local components of the linking numbers (3), we restrict to the nondegenerate case. First notice that

$$< \phi, \psi >_\gamma = \int_{\mathbf{T}(\mathbb{A}_{\mathbb{F},f})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{F},f})} \int_{\mathbf{T}(\mathbb{A}_{\mathbb{F},f})} \phi(t^{-1}\gamma t y)\ dt\ \bar{\psi}(y)\ dy. \tag{4}$$

Assume $\phi = \prod_v \phi_v$ and $\psi = \prod_v \psi_v$ are products of local components. Then

$$\int_{\mathbf{T}(\mathbb{A}_{\mathbb{F},f})} \phi(t^{-1}\gamma ty)\ dt \ = \ \prod_v \int_{\mathbf{T}(F_v)} \phi_v(t_v^{-1}\gamma_v t_v y_v)\ dt_v$$

as well as $<\phi,\psi>_\gamma = \prod_v <\phi,\psi>_{\gamma,v}$, where

$$(5) \qquad <\phi,\psi>_{\gamma,v} := \int_{\mathbf{T}(\mathbb{F}_v)\backslash\mathbf{G}(\mathbb{F}_v)} \int_{\mathbf{T}(\mathbb{F}_v)} \phi_v(t_v^{-1}\gamma_v t_v y_v)\ dt_v\ \bar{\psi}_v(y_v)\ dy_v.$$

Observe that $<\phi,\psi>_{\gamma,v}$ depends on the choice γ while $<\phi,\psi>_\gamma$ does not. An apropriate local definition is given below (Definition 2.1).

As all the following is local, we simplify notation: Let F denote a localization of $\mathbb{F}$ at a finite place not dividing 2. Let K be the quadratic extension of F given by localising $\mathbb{K}$. K is either a field, $K = F(\sqrt{A})$, or a split algebra $K = F \oplus F$. For $t \in K$, let $\bar{t}$ denote the Galois conjugate of t (resp. $\overline{(x,y)} = (y,x)$ in the split case). The local ring of F (resp. K) is $\mathbf{o}_F$ (resp. $\mathbf{o}_K$). It contains the maximal ideal $\wp_F$ (resp. $\wp_K$, where in the split case $\wp_K := \wp_F \oplus \wp_F$). Let π_F be a uniformizer for $\mathbf{o}_F$. If it can't be mixed up, we write $\wp$ (resp. π) for $\wp_F$ (resp. π_F). The residue class field of F has characteristic p and q elements. Further, let ω be the quadratic character of $F^\times$ given by the extension K/F that is, $\omega(x) = -1$ if x is not in the image of the the norm of K/F. Let $D := \mathbb{D}(F)$, $T := \mathbf{T}(F)$ and $G := \mathbf{G}(F)$. There exists $\epsilon \in D^\times$ such that for all $t \in K$ we have $\epsilon t = \bar{t}\epsilon$ and

$$D = K + \epsilon K.$$

Let $c := \epsilon^2 \in F^\times$. Let N denote the reduced norm on D. Restricted to K it equals the norm of K/F. For $\gamma_1, \gamma_2 \in K$ we have

$$\mathrm{N}(\gamma_1 + \epsilon\gamma_2) = \mathrm{N}(\gamma_1) - c\,\mathrm{N}(\gamma_2)\,.$$

D splits exactly in case $c \in \mathrm{N}(K^\times)$. We parametrize the double cosets $[\gamma] \in T\backslash G/T$ by the projective line:

Definition 2.1. Let $P : T\backslash G/T \to \mathbb{P}^1(F)$ be given by

$$P(\gamma_1 + \epsilon\gamma_2) \ := \ \frac{c\,\mathrm{N}(\gamma_2)}{\mathrm{N}(\gamma_1)}$$

for $\gamma_1 + \epsilon\gamma_2 \in D^\times$ as above.

This is well-defined: $P(t(\gamma_1 + \epsilon\gamma_2)t') = P(\gamma_1 + \epsilon\gamma_2)$ for all $t, t' \in K^\times$. The non-empty fibres of P not belonging to 0 or ∞ are exactly the nondegenerate double cosets. In case that K/F is a field extension, P is injective with range $c\,\mathrm{N}(K^\times) \cup \{0,\infty\}$. In case K/F split, the range of P is $F^\times\backslash\{1\} \cup \{0,\infty\}$ and the fibres of $F^\times\backslash\{1\}$ are single double cosets ([6]). This is one possible parametrization, another is $\xi := \frac{P}{P-1}$.

Lemma 2.2. *([13] Chapter 4) Let $\gamma \in D^\times$. In the double coset $T\gamma T$ of G there exists one and only one T-conjugacy class of trace zero.*

Now the local components $<\phi,\psi>_\gamma$ of the linking numbers can be declared precisely:

Definition 2.3. Let $\phi, \psi \in \mathcal{S}(\chi, G)$. For $x \in F^\times$ define the **local linking number**

$$< \phi, \psi >_x := < \phi, \psi >_{\gamma(x)}$$

if there is a trace zero preimage $\gamma(x) \in D^\times$ of x under P. If there is no preimage, let $< \phi, \psi >_x := 0$. Thus, for $x \in c\,\mathrm{N} := c\,\mathrm{N}(K^\times)$

$$< \phi, \psi >_x = \int_{T \backslash G} \int_T \phi(t^{-1}\gamma(x)ty)\; dt\; \bar{\psi}(y)\; dy.$$

By unimodularity of the Haar measure on T, this definition is independent of the choice of the element $\gamma(x)$ of trace zero. In all the following we make a general natural assumption on the character χ:

Hypothesis 2.4. *The conductors of χ and ω are coprime.*

The conductor $f(\chi) < \mathbf{o}_K$ of χ may be viewed as an ideal of $\mathbf{o}_F$: If $K = F \oplus F$, then $\chi = (\chi_1, \chi_1^{-1})$ for a character χ_1 of $F^\times$ and $f(\chi) = f(\chi_1)$. If K/F is a ramified field extension, then χ is unramified, thus $f(\chi) \cap \mathbf{o}_F = \mathbf{o}_F$. If K/F is an unramified field extension, then $f(\chi) = \pi^{c(\chi)}\mathbf{o}_K$, where π is an uniformizing element for K as well as F. That is, $f(\chi) \cap \mathbf{o}_F = \pi^{c(\chi)}\mathbf{o}_F$. There are some simple properties of χ following from the Hypothesis 2.4.

Lemma 2.5. *Assume 2.4. The following are equivalent:*
(a) χ is quadratic.
(b) χ factorizes via the norm.

Corollary 2.6. *Assume 2.4. If K/F is a ramified field extension, then χ is a quadratic character. If K/F is an unramified field extension and χ is unramified, then $\chi = 1$.*

We use compatible Haar measures: Let da be a nontrivial additive Haar measure on F. Then the measure $d^\times a$ of the multiplicative group $F^\times$ is normalized by

$$\mathrm{vol}^\times(\mathbf{o}_F^\times) = (1 - q^{-1})\,\mathrm{vol}(\mathbf{o}_F^\times)\,,$$

where vol (resp. $\mathrm{vol}^\times$) is the volume associated to da (resp. $d^\times a$). The measure on $T\backslash G$ is the quotient measure induced of those on G and T.

2.2. **Automorphic forms.** The central object on the analytic side is the Rankin-Selberg convolution of two automorphic representations. Gross-Zagier formulae describe the central order of its L-function.
Let Π_1 be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$ with trivial central character (i.e. an irreducible component of the discrete spectrum of the right translation on $L^2(\mathrm{GL}_2(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{F})), 1))$ and conductor N. Further, let $\Pi(\chi)$ be the irreducible component belonging to χ of the Weil representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$ for the norm form of $\mathbb{K}/\mathbb{F}$ (e.g. [2] §7). It has conductor $f(\chi)^2 f(\omega)$ and central character ω. The Rankin-Selberg convolution of Π_1 and $\Pi(\chi)$ produces ([5]) the Mellin transform

$$\Psi(s, W_1, W_2, \Phi) = \int_{Z(F)N(F)\backslash \mathrm{GL}_2(F)} W_1(g)W_2(eg)f_\Phi(s, \omega, g)\; dg$$

for Whittaker functions W_1 of Π_1 (resp. W_2 of $\Pi(\chi)$) for an arbitrary nontrivial character of F. Here $e := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. For a function $\Phi \in \mathcal{S}(F^2)$ let

$$f_\Phi(s,\omega,g) = |\det g|^s \int_{F^\times} \Phi\left((0,t)g\right) |t|^{2s} \omega(t)\ d^\times t\ .$$

f_Φ belongs to the principal series $\Pi(|\cdot|^{s-\frac{1}{2}}, \omega|\cdot|^{\frac{1}{2}-s})$. There is an adèlic analogue. Analytical continuation of Ψ leads to the L-function, the greatest common divisor of all Ψ. It is defined by newforms ϕ for Π_1 and θ_χ of $\Pi(\chi)$ as well as a special form E of $\Pi_E := \Pi(|\cdot|^{s-\frac{1}{2}}, \omega|\cdot|^{\frac{1}{2}-s})$:

$$\begin{aligned} L(s,\Pi_1\times\Pi(\chi)) &= \int_{Z(\mathbb{A}_\mathbb{F})\operatorname{GL}_2(\mathbb{F})\backslash \operatorname{GL}_2(\mathbb{A}_\mathbb{F})} \phi(g)\theta_\chi(g)E(s,g)\ dg \\ &= \int_{Z(\mathbb{A}_\mathbb{F})\operatorname{GL}_2(\mathbb{F})\backslash \operatorname{GL}_2(\mathbb{A}_\mathbb{F})} W_\phi(g)W_{\theta_\chi}(g)f_E(s,\omega,g)\ dg, \end{aligned}$$

where W_ϕ etc. denotes the associated Whittaker function. For places where $c(\chi)^2c(\omega) \leq v(N)$, the form E (resp. W_E) is the newform of the Eisenstein series. As Π_1 and $\Pi(\chi)$ are selfdual, the functional equation is

$$L(s,\Pi_1\times\Pi(\chi)) \ = \ \epsilon(s,\Pi_1\times\Pi(\chi))L(1-s,\Pi_1\times\Pi(\chi))\ .$$

In [13] (Chap. 1.4) an integral kernel $\Xi(s,g)$ is constructed which has a functional equation analogous to that of L and for which

$$L(s,\Pi_1\times\Pi(\chi)) = \int_{Z(\mathbb{A}_\mathbb{F})\operatorname{GL}_2(\mathbb{F})\backslash \operatorname{GL}_2(\mathbb{A}_\mathbb{F})} \phi(g)\Xi(s,g)\ dg.$$

We do not report the construction of this kernel, but we remark that the kernel depends on the newform of the theta series $\Pi(\chi)$ as well as the Eisenstein series Π_E, but not on the choice of Π_1. Its local nonconstant Fourier coefficients are defined by

$$W(s,\xi,\eta,g) := W_\theta(\begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} g) W_E(s, \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g). \tag{6}$$

Here $\eta := 1-\xi$. These Fourier coefficients are exactly those analytic functions which are compared to special local linking numbers in the local Gross-Zagier formula ([13] Lemma 4.3.1). We get ride of the restriction to newforms in (6) by reading it in the Kirillov models of the representations: Starting from the Whittaker model $\mathcal{W}(\Pi,\psi)$ of an irreducible admissible representation Π for an additive character ψ, the Kirillov space $\mathcal{K}(\Pi)$ is given by

$$\begin{aligned} \mathcal{W}(\Pi,\psi) &\to \mathcal{K}(\Pi), \\ W &\mapsto k : (a \mapsto W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Proposition 2.7. *([3], I.36) Let Π be an infinite dimensional irreducible admissible representation of $\operatorname{GL}_2(F)$. The Kirillov space $\mathcal{K}(\Pi)$ is generated by the Schwartz space $\mathcal{S}(F^\times)$ along with the following stalks around zero:*

(a) If Π is supercuspidal, this stalk is zero.
(b) If $\Pi = \Pi(\mu_1, \mu_2)$ is a principle series representation, then it is given by representatives of the form

- $\left(|a|^{\frac{1}{2}} c_1\mu_1(a) + |a|^{\frac{1}{2}} c_2\mu_2(a)\right) \mathbf{1}_{\wp^n}(a)$, *if* $\mu_1 \neq \mu_2$,
- $|a|^{\frac{1}{2}}\mu_1(a)\,(c_1 + c_2 v(x))\,\mathbf{1}_{\wp^n}(a)$, *if* $\mu_1 = \mu_2$.

Here $c_1, c_2 \in \mathbb{C}$.
(c) If $\Pi = \Pi(\mu_1, \mu_2)$ is special, it is given by representatives

- $|a|^{\frac{1}{2}}\mu_1(a)\mathbf{1}_{\wp^n}(a)$, *if* $\mu_1\mu_2^{-1} = |\cdot|$,
- $|a|^{\frac{1}{2}}\mu_2(a)\mathbf{1}_{\wp^n}(a)$, *if* $\mu_1\mu_2^{-1} = |\cdot|^{-1}$.

Definition 2.8. Let $\Pi(\chi)$ be the theta series and Π_E be the Eisenstein series at the central place $s = \frac{1}{2}$. The products

$$W(\xi, \eta) = W_\theta(\eta)W_E(\xi)$$

of Kirillov functions $W_\theta \in \mathcal{K}(\Pi(\chi))$ and $W_E \in \mathcal{K}(\Pi(1, \omega))$ are called **Whittaker products**. As $\eta = 1 - \xi$, they define linear forms on the isobaric sum $\Pi(\chi) \boxplus \Pi_E$. We denote the corresponding space of distributions by $\mathcal{W}$.

Being a component of a Weil representation, the theta series $\Pi(\chi)$ is completely described ([7] §1, [2] §7). Adèlically, it is a Hilbert modular form of conductor $f(\chi)^2 f(\omega)$ and of weight $(1, \ldots, 1)$ at the infinite places. If $K = F \oplus F$ is split, then $\chi = (\chi_1, \chi_1^{-1})$ and $\Pi(\chi) = \Pi(\chi_1, \omega\chi_1^{-1}) = \Pi(\chi_1, \chi_1^{-1})$ is a principle series representation. If K/F is a field extension and χ does not factorize via the norm, then $\Pi(\chi)$ is supercuspidal. While if $\chi = \chi_1 \circ \mathrm{N}$, it is the principle series representation $\Pi(\chi_1, \chi_1^{-1}\omega) = \Pi(\chi_1, \chi_1\omega)$, as $\chi_1^2 = 1$ by Lemma 2.5. Thus, by Proposition 2.7:

Proposition 2.9. *Let $\Pi(\chi)$ be the theta series and let $\mathcal{K}(\Pi(\chi))$ be its Kirillov space. It is a function space in one variable η generated by $\mathcal{S}(F^\times)$ along with the following stalks around zero:*

- *The zero function, if K/F is a field extension and $\chi \neq 1$.*
- $|\eta|^{\frac{1}{2}}\chi_1(\eta)\,(a_1 + a_2\omega(\eta))$, *if K/F is a field extension and $\chi^2 = 1$.*
- $|\eta|^{\frac{1}{2}}\left(a_1\chi_1(\eta) + a_2\chi_1^{-1}(\eta)\right)$, *if K/F is split and $\chi_1^2 \neq 1$,*
- $|\eta|^{\frac{1}{2}}\chi_1(\eta)\,(a_1 + a_2 v(\eta))$, *if K/F is split and $\chi_1^2 = 1$.*

We collect some properties of principal series. For an automorphic form $f \in \Pi(\mu_1|\cdot|^{s-\frac{1}{2}}, \mu_2|\cdot|^{\frac{1}{2}-s})$ there is $\Phi \in \mathcal{S}(F^2)$ such that

$$f(s, g) = \mu_1(\det g)|\det g|^s \int_{F^\times} \Phi\left((0, t)g\right)(\mu_1\mu_2^{-1})(t)|t|^{2s}\, d^\times t. \tag{7}$$

Conversely, any $\Phi \in \mathcal{S}(F^2)$ defines a form $f_\Phi \in \Pi(|\cdot|^{s-\frac{1}{2}}, \omega|\cdot|^{\frac{1}{2}-s})$ in that way ([1] Chap. 3.7). The Whittaker function belonging to f (in a Whittaker model with unramified character ψ) is given by the first Fourier coefficient

$$W_f(s, g, \psi) \;=\; \int_F f(s, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\bar{\psi}(x)\, dx.$$

Read off in the Kirillov model, the form for $s=\frac{1}{2}$ is given by evaluation at $g=\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, thus

$$W_f(a) \;:=\; W_f(\frac{1}{2}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \psi).$$

For unramified μ_i the newform is obtained by choosing

$$\phi(x,y) = \mathbf{1}_{\mathbf{o}_F}(x)\mathbf{1}_{\mathbf{o}_F}(y)$$

in (7). Thus,

$$\begin{aligned} W_{\mathrm{new}}(a) &= \mu_1(a)|a|^{\frac{1}{2}} \int_F \int_{F^\times} \mathbf{1}_{\mathbf{o}_F}(at)\mathbf{1}_{\mathbf{o}_F}(xt)\mu_1\mu_2^{-1}(t)|t| \; d^\times t \; \bar{\psi}(x) \; dx \\ &= \mu_1(a)|a|^{\frac{1}{2}} \mathbf{1}_{\mathbf{o}_F}(a) \operatorname{vol}(\mathbf{o}_F) \operatorname{vol}^\times(\mathbf{o}_F^\times) \sum_{j=-v(a)}^{0} \mu_1\mu_2^{-1}(\pi^j) \\ &= |a|^{\frac{1}{2}} \mathbf{1}_{\mathbf{o}_F}(a) \operatorname{vol}(\mathbf{o}_F) \operatorname{vol}^\times(\mathbf{o}_F^\times) \begin{cases} \frac{\mu_1(a\pi)-\mu_2(a\pi)}{\mu_1(\pi)-\mu_2(\pi)}, \text{ if } \mu_1 \neq \mu_2 \\ \mu_1(a)(v(a)+1), \text{ if } \mu_1 = \mu_2 \end{cases}. \end{aligned} \tag{8}$$

By Proposition 2.7 we have:

Proposition 2.10. *At $s=\frac{1}{2}$ the Eisenstein series Π_E is the principle series representation $\Pi(1,\omega)$. Its Kirillov space as a function space in the variable ξ is generated by $\mathcal{S}(F^\times)$ along with the following stalks around zero:*

- $|\xi|^{\frac{1}{2}}\,(a_1 + a_2\omega(\xi))$, *if K/F is a field extension,*
- $|\xi|^{\frac{1}{2}}\,(a_1 + a_2 v(\xi))$, *if K/F is split.*

For a finite set S of places of $\mathbb{F}$, let $\hat{\mathbf{o}}_{\mathbb{F}}^S := \prod_{v\notin S} \mathbf{o}_{\mathbb{F}_v}$ and $\mathbb{A}_S := \prod_{v\in S} \mathbb{F}_v \cdot \hat{\mathbf{o}}_{\mathbb{F}}^S$. We recall a property of Hecke operators.

Proposition 2.11. *([13] Chapter 2.4) Let μ be a character of $\mathbb{A}^\times/\mathbb{F}^\times$. Let $\phi \in L^2(\mathrm{GL}_2(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{A}), \mu)$, and let W_ϕ be the Whittaker function of ϕ in some Whittaker model. Let S be the finite set of infinite places and of those finite places v for which ϕ_v is not invariant under the maximal compact subgroup $\mathrm{GL}_2(\mathbf{o}_{\mathbb{F}_v})$. For $b \in \hat{\mathbf{o}}_{\mathbb{F}}^S \cap \mathbb{A}^\times$ define*

$$H(b) := \left\{ g \in M_2(\hat{\mathbf{o}}_{\mathbb{F}}^S) \mid \det(g)\hat{\mathbf{o}}_{\mathbb{F}}^S = b\hat{\mathbf{o}}_{\mathbb{F}}^S \right\}.$$

Then the following Hecke operator $\mathbf{T}_b$ is well defined for $g \in \mathrm{GL}_2(\mathbb{A}_S)$:

$$\mathbf{T}_b W_\phi(g) := \int_{H(b)} W_\phi(gh) \; dh.$$

If $y \in \hat{\mathbf{o}}_{\mathbb{F}}^S$ and $(b, y_f) = 1$, then

$$\mathbf{T}_b W_\phi(g \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) = |b|^{-1} W_\phi(g \begin{pmatrix} yb & 0 \\ 0 & 1 \end{pmatrix}).$$

That is, the action of the Hecke operator $\mathbf{T}_b$ on some Whittaker product is essentially translation by b:

$$\mathbf{T}_b W(\xi,\eta) = |b|^{-2} W(b\xi, b\eta). \tag{9}$$

3. Expansion of local linking numbers

Let $\mathcal{L}$ denote the space of distributions defined by the local linking numbers $<\phi,\psi>_x$ for $\phi,\psi\in\mathcal{S}(\chi,G)$. We desribe the expansion in $x\in F^\times$ of $\mathcal{L}$. The characterizing properties are close to those satisfied by the orbital integrals of [6], and Propositions 3.1 and 3.2 are influenced by the methods there. We distinguish whether the torus T is compact or not.

Proposition 3.1. *Let $K=F(\sqrt{A})$ be a field extension and let ω be the associated quadratic character. Let $\phi,\psi\in\mathcal{S}(\chi,G)$. The local linking number $<\phi,\psi>_x$ is a function of $x\in F^\times$ with the following properties:*
(a) It is zero on the complement of $c\,\mathrm{N}$.
(b) It is zero on a neighborhood of $1\in F^\times$.
(c) There is a locally constant function A_1 on a neighborhood U of 0 depending on ϕ such that for all $0\neq x\in U$: $<\phi,\psi>_x=A_1(x)(1+\omega(cx))$.
(d) There is an open set V containing zero such that for all $x^{-1}\in V\cap c\,\mathrm{N}$

$$<\phi,\psi>_x=\ \delta(\chi^2=1)\chi_1(\frac{A}{c})\chi_1(x)\int_{T\backslash G}\phi(\epsilon y)\bar{\psi}(y)\ dy.$$

Here the character χ_1 of $F^\times$ is given by $\chi=\chi_1\circ\mathrm{N}$ if $\chi^2=1$. Especially, the local linking number vanishes in a neighborhood of infinity if $\chi^2\neq 1$.

Proposition 3.2. *Let $K=F\oplus F$ be a split algebra. Let $\chi=(\chi_1,\chi_1^{-1})$ and let $\phi,\psi\in\mathcal{S}(\chi,G)$. The local linking number $<\phi,\psi>_x$ is a function of $x\in F^\times$ with the following properties:*
(a) It is zero on a neighborhood of $1\in F^\times$.
(b) It is locally constant on $F^\times$.
(c) There is an open set $U\ni 0$ and locally constant functions A_1,A_2 on U depending on ϕ and ψ such that for $0\neq x\in U$: $<\phi,\psi>_x=A_1(x)+A_2(x)v(x)$.
(d) There is an open set V containing zero and locally constant functions B_1,B_2 on V depending on ϕ and ψ such that for $x^{-1}\in V$:

$$<\phi,\psi>_x=\ \begin{cases}\chi_1(x)(B_1(x^{-1})+B_2(x^{-1})v(x)), & \text{if } \chi_1^2=1\\ \chi_1(x)B_1(x^{-1})+\chi_1^{-1}(x)B_2(x^{-1}), & \text{if } \chi_1^2\neq 1\end{cases}.$$

For $\chi_1^2=1$, the function B_2 is nonzero only if $\mathrm{id}\in\operatorname{supp}\phi(\operatorname{supp}\psi)^{-1}$.

We need two lemmas.

Lemma 3.3. *Let $\phi\in\mathcal{S}(\chi,G)$.*
(a) For each $y\in G$ there is an open set $V\ni y$ such that for all $g\in\operatorname{supp}(\phi)y^{-1}$ and all $\tilde{y}\in V$

$$\phi(g\tilde{y})\ =\ \phi(gy).$$

(b) Let $C\subset G$ be compact. For each $g\in G$ there is an open set $U\ni g$ such that for all $\tilde{g}\in U$ and all $y\in TC$

$$\int_T\phi(t^{-1}\tilde{g}ty)\ dt\ =\ \int_T\phi(t^{-1}gty)\ dt. \tag{10}$$

Proof of Lemma 3.3. (a) It is enough to prove the statement for $y = \mathrm{id}$. As ϕ is locally constant, for every $g \in G$ there is an open set $U_g \ni \mathrm{id}$ with $\phi(gU_g) = \phi(g)$. Let $C \subset G$ be compact such that $\operatorname{supp}\phi = TC$. We cover C by finitely many gU_g and choose U to be the intersection of those U_g. Then $\phi(gU) = \phi(g)$ for all $g \in TC$.
(b) It is enough to prove the statement for $y \in C$ rather than $y \in TC$, as a factor $s \in T$ just changes the integral by a factor $\chi(s)$. By (a) there is an open set $V_y \ni y$ such that $\phi(t^{-1}gt\tilde{y}) = \phi(t^{-1}gty)$ for $\tilde{y} \in V_y$ and $t^{-1}gt \in \operatorname{supp}(\phi)y^{-1}$. Take finitely many $y \in C$ such that the V_y cover C. It is enough to find open sets $U_y \ni g$ for these y so that Equation (10) is fulfilled. Then $\cap U_y$ is an open set such that (10) is satisfied for all $y \in TC$. Write $g = g_1 + \epsilon g_2$ and describe a neighborhood U_y of g by $k_1, k_2 > 0$ depending on y and the obstructions $|\tilde{g}_i - g_i| < k_i$, $i = 1, 2$, for $\tilde{g}$ lying in U_y. Write $t^{-1}\tilde{g}t = g_1 + \epsilon g_2 t\bar{t}^{-1} + (\tilde{g}_1 - g_1) + \epsilon(\tilde{g}_2 - g_2)t\bar{t}^{-1}$. As ϕ is locally constant, we may choose k_1, k_2 depending on y such that

$$\phi(t^{-1}\tilde{g}t) \;=\; \phi((g_1 + \epsilon g_2 t\bar{t}^{-1})y) = \phi(t^{-1}gty).$$

These constants are independent from t as $|(\tilde{g}_2 - g_2)t\bar{t}^{-1}| = |\tilde{g}_2 - g_2|$. □

Lemma 3.4. *Let $\phi \in \mathcal{S}(F \oplus F)$.*
(a) There are $A_1, A_2 \in \mathcal{S}(F)$ such that

$$\int_{F^\times} \phi(a^{-1}y, a)\; d^\times a \;=\; A_1(y) + A_2(y)v(y).$$

(b) Let η be a nontrivial (finite) character of $F^\times$. There are $B_1, B_2 \in \mathcal{S}(F)$ and $m \in \mathbb{Z}$ such that for $0 \neq y \in \wp^m$

$$\int_{F^\times} \phi(a^{-1}y, a)\eta(a)\; d^\times a \;=\; B_1(y) + B_2(y)\eta(y).$$

Proof of Lemma 3.4. (a) Any $\phi \in \mathcal{S}(F \oplus F)$ is a finite linear combination of the following elementary functions: $\mathbf{1}_{\wp^n}(a)\mathbf{1}_{\wp^n}(b)$, $\mathbf{1}_{x+\wp^n}(a)\mathbf{1}_{\wp^n}(b)$, $\mathbf{1}_{\wp^n}(a)\mathbf{1}_{z+\wp^n}(b)$, $\mathbf{1}_{x+\wp^n}(a)\mathbf{1}_{z+\wp^n}(b)$ for suitable $n \in \mathbb{Z}$ and $v(x), v(z) > n$. It is enough to prove the statement for these functions. We get

$$\int_{F^\times} \mathbf{1}_{\wp^n}(a^{-1}y)\mathbf{1}_{\wp^n}(a)\; d^\times a \;=\; \mathbf{1}_{\wp^{2n}}(y)v(y\pi^{-2n+1})\operatorname{vol}^\times(\mathbf{o}_F^\times).$$

Thus, if $0 \in \operatorname{supp}\phi$ the integral has a pole at $y = 0$, otherwise it hasn't:

$$\int_{F^\times} \mathbf{1}_{x+\wp^n}(a^{-1}y)\mathbf{1}_{\wp^n}(a)\; d^\times a \;=\; \mathbf{1}_{\wp^{v(x)+n}}(y)\operatorname{vol}^\times(1 + \wp^{n-v(x)}),$$

$$\int_{F^\times} \mathbf{1}_{\wp^n}(a^{-1}y)\mathbf{1}_{z+\wp^n}(a)\; d^\times a \;=\; \mathbf{1}_{\wp^{v(z)+n}}(y)\operatorname{vol}^\times(1 + \wp^{n-v(z)})$$

and

$$\int_{F^\times} \mathbf{1}_{x+\wp^n}(a^{-1}y)\mathbf{1}_{z+\wp^n}(a)\; d^\times a \;=\; \mathbf{1}_{xz(1+\wp^m)}(y)\operatorname{vol}^\times(1 + \wp^m),$$

where $m := n - \min\{v(x), v(z)\}$.
(b) Similar computations to those of (a). □

Proof of Proposition 3.1. (a) is clear by definition.
(b) Assume $1 \in c\,\mathrm{N}$, otherwise this property is trivial. We have to show that for all γ with $P(\gamma) \in U$, where U is a sufficiently small neighborhood of 1,

$$\int_{T\backslash G}\int_T \phi(t^{-1}\gamma ty)\ dt\bar{\psi}(y)\ dy = 0.$$

We show that the inner integral is zero. Let $C \subset G$ be compact such that $\operatorname{supp}\phi \subset TC$. Then ϕ vanishes outside of TCT. It is enough to show that there is $k > 0$ such that $|P(\gamma)-1| > k$ holds for all $\gamma \in TCT$. Assume there isn't such k. Let $(\gamma_i)_i$ be a sequence such that $P(\gamma_i)$ tends to 1. Multiplying by elements of T and enlarging C occasionally (this is possible as T is compact), we assume $\gamma_i = 1 + \epsilon t_i = z_i c_i$, where $t_i \in T$, $c_i \in C$, $z_i \in Z$. Then $P(\gamma_i) = ct_i\bar{t}_i = 1 + a_i$, where $a_i \to 0$. We have $\det\gamma_i = 1 - ct_i\bar{t}_i = -a_i$ as well as $\det\gamma_i = z_i^2\det c_i$. As C is compact, $(z_i)_i$ is forced to tend to zero. This implies $\gamma_i \to 0$ contradicting $\gamma_i = 1 + \epsilon t_i$.
(c) A coset $\gamma \in F^\times\backslash D^\times$ of trace zero has a representative of the form $\gamma = \sqrt{A} + \epsilon\gamma_2$ (by abuse of notation). Thus,

$$<\phi,\psi>_x = \int_{T\backslash G}\int_T \phi((\sqrt{A}+\epsilon\gamma_2 t\bar{t}^{-1})y)\ dt\ \bar{\psi}(y)\ dy.$$

As $\phi \in \mathcal{S}(\chi, G)$, there exists an ideal $\wp_K^m$ of K such that for all $y \in G$ and all $l \in \wp_K^m$ one has $\phi((\sqrt{A}+\epsilon l)y) = \phi(\sqrt{A}y)$. Let $x = P(\gamma)$ be near zero, i.e. x belongs to an ideal U of F given by the obstruction that $\frac{cl\bar{l}}{-A} \in U$ implies $l \in \wp_K^m$. For such x we have

$$<\phi,\psi>_x = \operatorname{vol}_T(T)\chi(\sqrt{A})\int_{T\backslash G}\phi(y)\bar{\psi}(y)\ dy.$$

So if $\langle\cdot,\cdot\rangle$ denotes the L^2-scalar product, we have

$$<\phi,\psi>_x = \frac{1}{2}\operatorname{vol}_T(T)\chi(\sqrt{A})\langle\phi,\psi\rangle(1+\omega(cx)).$$

(d) Let $\gamma = \sqrt{A} + \epsilon\gamma_2$ denote a trace zero preimage of x under P. Then

$$\int_T \phi(t^{-1}\gamma ty)\ dt\ =\ \chi(\gamma_2)\int_T \phi((\sqrt{A}\gamma_2^{-1} + t^{-1}\bar{t}\epsilon)y)\ dt.$$

By Lemma 3.3 there exists $k > 0$ such that for $|\gamma_2| > k$ and for $y \in \operatorname{supp}\psi$ we have $\phi((\sqrt{A}\gamma_2^{-1} + t^{-1}\bar{t}\epsilon)y) = \phi(t^{-1}\bar{t}\epsilon y)$. Thus, for $|x| > |cA^{-1}|k^2$,

$$<\phi,\psi>_x = \chi(\gamma_2)\int_T \chi(t^{-1}\bar{t})\ dt\int_{T\backslash G}\phi(\epsilon y)\bar{\psi}(y)\ dy.$$

As $\chi(t^{-1}\bar{t})$ defines the trivial character of T if and only if $\chi^2 = 1$, the statement follows by noticing that in this case $\chi(\gamma_2) = \chi_1(\frac{Ax}{c})$. $\square$

Proof of Proposition 3.2. There is an isomorphism from $D^\times$ to $\mathrm{GL}_2(F)$ given by embedding $K^\times$ diagonally and sending ϵ to $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Then P is given by

$$P\begin{pmatrix}a&b\\c&d\end{pmatrix}=\frac{bc}{ad}.$$

The only value not contained in the image of P is 1. A preimage of $x\neq 1$ of trace zero is

$$\gamma(x) \;=\; \begin{pmatrix}-1&x\\-1&1\end{pmatrix}.$$

(a) We show that for $\phi\in\mathcal{S}(\chi,G)$ there is a constant $k>0$ such that for all $\gamma\in\operatorname{supp}\phi$: $|P(\gamma)-1|>k$. By Bruhat-Tits decomposition, $G=\mathrm{PGL}_2(F)=TNN'\cup TNwN$, where N is the group of uniponent upper triangular matrices, N' its transpose and $w=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Thus, there is $c>0$ such that

$$\begin{aligned}\operatorname{supp}\phi \;&\subset\; T\left\{\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}1&0\\v&1\end{pmatrix}\mid |u|<c,|v|<c\right\}\\ &\bigcup T\left\{\begin{pmatrix}1&u\\0&1\end{pmatrix}w\begin{pmatrix}1&v\\0&1\end{pmatrix}\mid |u|<c,|v|<c\right\}.\end{aligned}$$

On the first set P we have $P=\frac{uv}{1+uv}$. On the second one we have $P=\frac{uv-1}{uv}$. Thus, for all $\gamma\in\operatorname{supp}\phi$ we have $|P(\gamma)-1|\geq\min\{1,c^{-2}\}$. Next we show that there is a constant $k>0$ such that $|P(\gamma)-1|>k$ for all $\gamma y\in\operatorname{supp}\phi$ for all $y\in\operatorname{supp}\psi$. This implies that $<\phi,\psi>_x=0$ in the neighborhood $|x-1|<k$ of 1. There is such a constant k_y for any $y\in\operatorname{supp}\psi$. By Lemma 3.3(a) this constant is valid for all $\tilde y$ in a neighborhood V_y. Modulo T the support of ψ can be covered by finitely many V_y. The minimum of the associated k_y is the global constant we claimed.
(b) By Lemma 3.3(b), there is for every $x\in F^\times\backslash\{1\}$ a neighborhood U_x such that for all $y\in\operatorname{supp}\psi$ the inner integral

$$\int_T\phi(t^{-1}\gamma(\tilde x)ty)\;dt$$

is constant in $\tilde x\in U_x$. Even more the local linking number is locally constant on $F^\times\backslash\{1\}$. By (a) it is locally constant in $x=1$ as well.
For (c) and (d) we regard the inner integral separately first. For representatives we have

$$\begin{aligned}t^{-1}\gamma(x)t \;&=\; \begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}\begin{pmatrix}-1&x\\-1&1\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}\\ &=\begin{pmatrix}(x-1)&0\\0&1\end{pmatrix}\begin{pmatrix}1&\frac{x}{a(x-1)}\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-a&1\end{pmatrix}\in K^\times NN'\\ &=\begin{pmatrix}\frac{1-x}{a}&0\\0&-a\end{pmatrix}\begin{pmatrix}1&\frac{a}{x-1}\\0&1\end{pmatrix}w\begin{pmatrix}1&-a^{-1}\\0&1\end{pmatrix}\in K^\times NwN.\end{aligned}$$

As $\operatorname{supp}\phi$ is compact modulo T, the intersections $\operatorname{supp}\phi \cap NN'$ and $\operatorname{supp}\phi \cap NwN$ are compact. We write ϕ^y for the right translation of ϕ by y. Then ϕ^y is a sum $\phi^y = \phi_1^y + \phi_2^y$, $\phi_i^y \in \mathcal{S}(\chi, G)$, with $\operatorname{supp}\phi_1^y \subset TNN'$ and $\operatorname{supp}\phi_2^y \subset TNwN$. Using the transformation under T by χ, we can actually regard ϕ_i^y, $i = 1, 2$, as functions on $F \oplus F$ identifying N with F. Thus, $\phi_i^y \in \mathcal{S}(F \oplus F)$. Then we get

$$\begin{aligned} (11) \quad \int_T \phi(t^{-1}\gamma(x)ty)\, dt &= \chi_1(x-1) \int_{F^\times} \phi_1^y(\frac{x}{a(x-1)}, -a)\, d^\times a \\ &+ \chi_1(1-x) \int_{F^\times} \chi_1(a^{-2}) \phi_2^y(\frac{a}{x-1}, -a^{-1})\, d^\times a. \end{aligned}$$

(c) We have $\chi_1(x-1) = \chi_1(-1)$ if $x \in \wp^{c(\chi_1)}$, where $c(\chi_1)$ is the leader of χ_1. By lemma 3.4, the first integral of (11) for small x equals

$$A_1(\frac{x}{x-1}) + A_2(\frac{x}{x-1}) v(\frac{x}{x-1}),$$

where A_1, A_2 are locally constant functions on a neighborhood of zero depending on y. Then $\tilde{A}_i(x) := A_i(\frac{x}{x-1})$ are locally constant functions on a neighborhood U_1 of zero as well. The second integral of (11) is constant on a neighborhood U_2 of $x = 0$ depending on y, as ϕ_2^y is locally constant for $(x-1)^{-1} \to -1$. Thus, the complete inner integral can be expressed on $U_y := \wp^{c(\chi_1)} \cap U_1 \cap U_2$ as

$$A_y(x) := \tilde{A}_1(x) + \tilde{A}_2(x) v(x) + B.$$

By lemma 3.3(a), there is a neighborhood V_y of y where the inner integral is constant. Take V_y that small that ψ is constant there, too, and cover $\operatorname{supp}\psi$ modulo T by finitely many such V_y, $y \in I$, for some finite set I. The local linking number for $x \in U = \cap_{y \in I} U_y$ now is computed as

$$< \phi, \psi >_x = \sum_{y \in I} \operatorname{vol}_{T\backslash G}(TV_y) \bar{\psi}(y) A_y(x).$$

That is, there are locally constant functions B_1, B_2 on U such that for $x \in U$

$$< \phi, \psi >_x = B_1(x) + B_2(x) v(x).$$

(d) Let $x^{-1} \in \wp^{c(\chi_1)}$. Then $\chi_1(x-1) = \chi(x)$. As ϕ_1^y is locally constant, the first integral of (11) equals a locally constant function $A_1(x^{-1})$ for x^{-1} in a neighborhood U_1 of zero depending on y. For the second integral, we distinguish whether $\chi_1^2 = 1$ or not. Let $\eta := \chi_1^2 \neq 1$. Applying lemma 3.4(b), we get locally constant functions A_2, A_3 on a neigborhood U_2 of zero depending on y such that the second integral equals $A_2(x^{-1}) + \chi_1^2(x^{-1}) A_3(x^{-1})$. Thus, for fixed y the inner integral for $x^{-1} \in U_y = U_1 \cap U_2 \cap \wp^{c(\chi_1)}$ is

$$A_y(x) := \int_T \phi^y(t^{-1}\gamma(c)t)\, dt = \chi_1(x)\big(A_1(x^{-1}) + A_2(x^{-1}) + A_3(x^{-1})\chi_1^{-1}(x)\big).$$

Proceeding as in (c), we get the assertion. Let $\chi_1^2 = 1$. By lemma 3.4(a), we have locally constant functions A_2, A_3 on a neighborhood U_2 of zero such that for $x^{-1} \in U$ the second integral of (11) is given by $A_2(x^{-1}) + A_2(x^{-1})v(x)$. Thus, for $x^{-1} \in U_y := U_1 \cap U_2 \cap \wp^{c(\chi_1)}$ the inner integral is given by

$$A_y(x) := \chi_1(x)\big(A_1(x^{-1}) + A_2(x^{-1}) + A_3(x^{-1})v(x)\big).$$

The term $A_3(x^{-1})v(x)$ by lemma 3.4(a) is obtainted from functions $\phi_2^y(a,b)$ having the shape $\mathbf{1}_{\wp^n}(a)\mathbf{1}_{\wp^n}(b)$ around zero. Those function can only occur if y is contained in $\operatorname{supp}\phi$. Again proceeding as in part (c), the local linking number for x^{-1} in a sufficently small neighborhood U of zero is

$$<\phi,\psi>_x = \chi_1(x)\big(B_1(x^{-1}) + B_2(x^{-1})v(x)\big),$$

where B_1, B_2 are locally constant on U and B_2 doesn't vanish only if $\mathrm{id} \in (\operatorname{supp}\phi)(\operatorname{supp}\psi)^{-1}$. □

The above properties of the local linking numbers describe them completely:

Proposition 3.5. *The properties (a) to (d) of Proposition 3.1 resp. 3.2 characterize $\mathcal{L}$: Given a function H on $F^\times$ satisfying these properties, there are $\phi,\psi \in \mathcal{S}(\chi,G)$ such that $H(x) = <\phi,\psi>_x$.*

We first describe the construction in general before going into detail in the case of a field extension K/F. The case of a split algebra $K = F \oplus F$ will be omitted, as it is similar and straightforward after the case of a field extension. A complete proof can be found in [9], Chapter 2. We choose a describtion of a function H satisfying the properties (a) to (d),

$$H(x) = \mathbf{1}_{c\,\mathrm{N}}(x)\big(A_0(x)\mathbf{1}_{V_0}(x) + A_1(x)\mathbf{1}_{V_1}(x) + \sum_{j=2}^{M} H(x_j)\mathbf{1}_{V_j}(x)\big),$$

where $V_j = x_j(1+\wp_F^{n_j})$, $j = 2,\dots,M$, are open sets in $F^\times$ on which H is constant. Similarly,

$$V_0 = \wp_F^{n_0} \quad \text{resp.} \quad V_1 = F\backslash\wp^{-n_1}$$

are neighborhoods of 0 (resp. ∞) where H is characterized by A_0 (resp. A_1) according to property (c) (resp. (d)). We may assume $n_j > 0$ for $j = 0,\dots,M$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Then we construct a function ψ and functions ϕ_j, $j = 0,\dots,M$, in $\mathcal{S}(\chi,G)$ such that $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j = \emptyset$ if $i \neq j$ and such that

$$<\phi_j,\psi>_x = H(x_j)\mathbf{1}_{V_j}(x) \quad \text{resp.} \quad <\phi_j,\psi>_x = A_j(x)\mathbf{1}_{V_j}(x).$$

There is essentially one possibility to construct such functions in $\mathcal{S}(\chi,G)$: Take a compact open subset C of G which is **fundamental** for χ, that is if $t \in T$ and $c \in C$ as well as $tc \in C$, then $\chi(t) = 1$. Then the function $\phi = \chi\cdot\mathbf{1}_C$ given by $\phi(tg) = \chi(t)\mathbf{1}_C(g)$ is well defined in $\mathcal{S}(\chi,G)$ with support TC. The function ψ is then chosen as $\psi = \chi\cdot\mathbf{1}_U$, where U is a compact open subgroup of G that small that for $j = 0,\dots,M$

$$P(P^{-1}(V_j)U) = V_j \cap c\,\mathrm{N}\,.$$

For $j \geq 2$ we take C_j compact such that C_jU is fundamental and $P(C_jU) = V_j$ and define $\phi_j := H(x_j)\cdot\chi\cdot\mathbf{1}_{C_jU}$. The stalks of zero and infinity are constructed similarly. As the local linking numbers are linear in the first component and as the supports of the ϕ_j are disjoint by construction, we get

$$H(x) \ = < \sum_{j=0}^{M} \phi_j, \psi >_x .$$

Proof of Proposition 3.5 in the case K a field. Let $K = F(\sqrt{A})$. Let the function H satisfying (a) to (d) of Prop. 3.1 be given by

$$H(x) = \mathbf{1}_{c\,\mathrm{N}}(x)\big(A_0(x)\mathbf{1}_{V_0}(x) + A_1(x)\mathbf{1}_{V_1}(x) + \sum_{j=2}^{M} H(x_j)\mathbf{1}_{V_j}(x)\big),$$

where

$$\begin{aligned} &V_0 = \wp^{n_0} \text{ and } A_0(x) = a_0,\\ &V_1 = F\backslash\wp^{-n_1} \text{ and } A_1(x) = \begin{cases} \chi_1(x)a_1, & \text{if } \chi^2 = 1\\ 0, & \text{if } \chi^2 \neq 1 \end{cases},\\ &V_j = x_j(1+\wp^{n_j}) \text{ for } j = 2,\dots,M, \end{aligned}$$

with $a_0, a_1, H(x_j) \in \mathbb{C}$, and $n_j > 0$ for $j = 0,\dots,M$. We further assume

$$n_0 - v(\frac{c}{A}) > 0, \quad n_1 + v(\frac{c}{A}) > 0 \text{ and both even,}$$

as well as $V_i \cap V_j = \emptyset$ for $i \neq j$. Let

$$\tilde{n}_0 \ = \ \begin{cases} \frac{1}{2}(n_0 - v(\frac{c}{A})), & \text{if } K/F \text{ unramified}\\ n_0 - v(\frac{c}{A}), & \text{if } K/F \text{ ramified} \end{cases},$$

$$\tilde{n}_1 \ = \ \begin{cases} \frac{1}{2}(n_1 + v(\frac{c}{A})), & \text{if } K/F \text{ unramified}\\ n_1 + v(\frac{c}{A}), & \text{if } K/F \text{ ramified} \end{cases},$$

as well as for $j = 2,\dots,M$

$$\tilde{n}_j = \begin{cases} n_j, & \text{if } K/F \text{ unramified}\\ 2n_j, & \text{if } K/F \text{ ramified} \end{cases}.$$

Then $\mathrm{N}(1+\wp_K^{\tilde{n}_j}) = 1 + \wp_F^{n_j}$, $j \geq 2$. Here $\wp_K$ is the prime ideal of K. Define

$$U := 1 + \wp_K^k + \epsilon\wp_K^m,$$

where $k > 0$ and $m > 0$ are chosen such that

$$\begin{aligned} &k \geq c(\chi), \quad m \geq c(\chi)\\ &k \geq \tilde{n}_j, \quad m \geq \tilde{n}_j + 1, \text{ for } j = 0,\dots,M,\\ &m \geq c(\chi) + 1 - \frac{1}{2}v(x_j), \text{ for } j = 2,\dots,M, \end{aligned}$$

$$m \geq \tilde{n}_j + 1 + \frac{1}{2}|v(x_j)|, \text{ for } j = 2,\dots,M. \tag{12}$$

As $k, m > 0$ and $k, m \geq c(\chi)$, U is fundamental. Define

$$\psi := \chi \cdot \mathbf{1}_U.$$

To realize the stalks for x_j, $j \geq 2$, let $\sqrt{A}+\epsilon\gamma_j$ be a preimage of x_j,

$$P(\sqrt{A}+\epsilon\gamma_j) \;=\; \frac{c\,\mathrm{N}(\gamma_j)}{-A} \;=\; x_j.$$

The preimage of V_j is

$$P^{-1}(V_j) \;=\; T\big(\sqrt{A}+\epsilon\gamma_j(1+\wp_K^{\tilde{n}_j})\big)T \;=\; T\big(\sqrt{A}+\epsilon\gamma_j(1+\wp_K^{\tilde{n}_j})\,\mathrm{N}_K^1\big).$$

Let $C_j := \sqrt{A}+\epsilon\gamma_j(1+\wp_K^{\tilde{n}_j})\,\mathrm{N}_K^1$. The compact open set

$$C_jU \;=\; \sqrt{A}(1+\wp_K^k)+c\bar{\gamma}_j\wp_K^m+\epsilon\big(\gamma_j(1+\wp_K^k+\wp_K^{\tilde{n}_j})\,\mathrm{N}_K^1+\sqrt{A}\wp_K^m\big).$$

is fundamental, due to the choices (12): We have to check that if $t\in T$, $c\in C_j$, and $tc\in C_jU$, then $\chi(t)=1$ (observe that U is a group). Let

$$tc \;=\; t\sqrt{A}+\epsilon\bar{t}\gamma_j(1+\pi_K^{\tilde{n}_j}c_1)l \;\in\; C_jU.$$

The first component forces $t\in 1+\wp_K^k+\frac{c}{A}\bar{\gamma}_j\wp_K^m$, for which $\chi(t)=1$, by (12). For the image of C_jU we find again by (12)

$$P(C_jU) \;=\; \frac{c\,\mathrm{N}(\gamma_j)\,\mathrm{N}(1+\wp_K^k+\wp_K^{\tilde{n}_j}+\wp_K^m\frac{\sqrt{A}}{\gamma_j})}{-A\,\mathrm{N}(1+\wp_K^k+\frac{c}{\sqrt{A}}\bar{\gamma}_j\wp_K^m)} \;=\; V_j.$$

So the functions $\phi_j := \chi\cdot\mathbf{1}_{C_jU}\in\mathcal{S}(\chi,G)$ are well defined. We compute

$$<\phi_j,\psi>_x = \int_{T\backslash G}\int_T \phi_j(t^{-1}\gamma(x)ty)\;dt\;\bar{\psi}(y)\;dy.$$

The integrand doesn't vanish only if there is $s\in K^\times$ such that

$$st^{-1}\gamma(x)t \;=\; s\sqrt{A}+\epsilon\bar{s}\gamma_2(x)t\bar{t}^{-1} \;\in\; C_jU.$$

The first component implies $s\in 1+\wp_K^{\tilde{n}_j}$. The second one implies $\gamma_2(x)\in\gamma_j(1+\wp_K^{\tilde{n}_j})\,\mathrm{N}_K^1$, which is equivalent to $x\in V_j$. In this case we take $s=1$ and get

$$<\phi_j,\psi>_x = \;\mathbf{1}_{V_j}(x)\int_{T\backslash G}\int_T 1\;dt\;\bar{\psi}(y)\;dy = \mathbf{1}_{V_j}(x)\,\mathrm{vol}_T(T)\,\mathrm{vol}_G(U).$$

Normalizing $\tilde{\phi}_j := \frac{H(x_j)}{\mathrm{vol}_T(T)\,\mathrm{vol}_G(U)}\phi_j$, we get $H|_{V_j}(x) =<\tilde{\phi}_j,\psi>_x$.
For the stalk at zero, we find $P(C_0)=\wp_F^{n_0}\cap c\,\mathrm{N}$, where $C_0 := \sqrt{A}+\epsilon\wp_K^{\tilde{n}_0}$. The preimage $P^{-1}(V_0)$ equals $TC_0T=TC_0$. The open and compact set C_0U is easily seen to be fundamental and to satisfy $P(C_0U)=V_0\cap cN$. Define $\phi_0 := \chi\cdot\mathbf{1}_{C_0U}$ and compute the local linking number $<\phi_0,\psi>_x$. It doesn't vanish only if there is $s\in K^\times$ such that

$$st^{-1}\gamma(x)t \;=\; s\sqrt{A}+\epsilon\bar{s}\gamma_2(x)t\bar{t}^{-1} \;\in\; C_0U.$$

This forces $\gamma_2(x)\in\wp_K^{\tilde{n}_0}$. Then we take $s=1$ and get

$$<\phi_0,\psi>_x = \;\mathbf{1}_{V_0\cap c\,\mathrm{N}}(x)\,\mathrm{vol}_T(T)\,\mathrm{vol}_G(U).$$

That is, $H|_{V_0}(x) = a_0 =< \frac{a_0}{\mathrm{vol}_T(T)\,\mathrm{vol}_G(U)}\phi_0, \psi >_x$. It remains to construct the stalk at infinity in case $\chi^2 = 1$. Thus, $\chi = \chi_1 \circ \mathrm{N}$. The preimage of $V_1 = F\backslash\wp_F^{-n_1}$ is given by

$$P^{-1}(V_1) \;=\; T\big(\sqrt{A} + \epsilon(\wp_K^{\tilde{n}_1})^{-1}\big)T = T\big(\sqrt{A}\wp_K^{\tilde{n}_1} + \epsilon\,\mathrm{N}_K^1\big).$$

Take $C_1 = \sqrt{A}\wp_K^{\tilde{n}_1} + \epsilon\,\mathrm{N}_K^1$ to get a fundamental compact open set

$$C_1U \;=\; \sqrt{A}\wp_K^{\tilde{n}_1} + c\wp_K^m + \epsilon\big(\mathrm{N}_K^1(1+\wp_K^k) + \sqrt{A}\wp_K^{m+\tilde{n}_1}\big),$$

By the choices (12) we get $P(C_1U) = V_1 \cap c\,\mathrm{N}$. Taking $\phi_1 := \chi \cdot \mathbf{1}_{C_1U}$ this time, we get $H|_{V_1}(x) = \frac{a_1}{\mathrm{vol}_T(T)\,\mathrm{vol}_G(U)} < \phi_1, \psi >_x$. □

We use the parametrization $\xi = \frac{x}{x-1}$. The properties of the local linking numbers (Propositions 3.1 and 3.2) transform accordingly. Let $\tilde{\mathcal{L}}$ be the space of distributions made up by evaluating the mutiple $|\xi\eta|^{\frac{1}{2}} < \phi, \psi >_{\gamma(\xi)}$ of local linking numbers at $\xi \in F^\times$ for $\phi, \psi \in \mathcal{S}(\chi, G)$. This is the space of test vectors of the geometric side, while the space $\mathcal{W}$ of analytic test vetors is given by evaluation of Whittaker products. We have the following transfer:

Theorem 3.6. *Assume $\omega(-\xi\eta) = 1$ if D is split, resp. $\omega(-\xi\eta) = -1$ if D is a division algebra. The spaces of test vectors $\tilde{\mathcal{L}}$ and $\mathcal{W}$ have identical ξ-expansion.*

Proof of Theorem 3.6. The space $\mathcal{W}$ is characterized by Propositions 2.9 and 2.10. Comparing it with $\tilde{\mathcal{L}}$ (Propositions 3.1 resp. 3.2) yields the claim. For example, by Prop. 2.9 for K/F split and $\chi_1^2 \neq 1$, the Whittaker products for $\xi \to 1$ ($\eta \to 0$) are given by

$$|\xi\eta|^{\frac{1}{2}}\left(a_1\chi_1(\eta) + a_2\chi_1^{-1}(\eta)\right),$$

which corresponds to Prop. 3.2 (d). For $\xi \to 0$ ($\eta \to 1$) we apply Prop. 2.10: The Whittaker products have the shape $|\xi\eta|^{\frac{1}{2}}(a_1 + a_2 v(\xi))$. This is property (c) of Prop. 3.2. Away from $\xi \to 1$ and $\xi \to 0$, the Whittaker products are locally constant with compact support. This is equivalent to (a) and (b) of Prop. 3.2. □

4. Translated linking numbers

In the remaining, the quaternion algebra D is assumed to be split, that is $G = F^\times\backslash D^\times$ is isomorphic to the projective group $\mathrm{PGL}_2(F)$. The aim is to give an operator on the local linking numbers realizing the Hecke operator on the analytic side. As the analytic Hecke operator essentially is given by translation by $b \in F^\times$ (Proposition 2.11), the first candidate for this study is the translation by b,

$$< \phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \psi >_x = \int_{T\backslash G} \int_T \phi(t^{-1}\gamma(x)ty)\;dt\;\bar{\psi}(y\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix})\;dy.$$

Let

$$I_\phi(y) \;=\; \int_T \phi(t^{-1}\gamma(x)ty)\;dt \tag{13}$$

be its inner integral. Here the difference between the case of a compact torus and that of a noncompact one becomes crucial. Fixing x and viewing the translated linking number as a function of b alone, we describe the behavior in the compact case in a few lines. In the noncompact case our computational approach makes up one hundred pages. Refering to [9], this case is only sketched.

4.1. **The compact case.** Let $K = F(\sqrt{A})$ be a field extension of F. So T is compact. As functions $\phi \in \mathcal{S}(\chi, G)$ have compact support modulo T, the set $T\gamma(x)T \cdot \operatorname{supp}\phi$ is compact. Left translation by $t' \in T$ yields $I_\phi(t'y) = \chi(t')I_\phi(y)$. Thus, the inner integral I_ϕ itself is an element of $\mathcal{S}(\chi, G)$. Choose the following isomorphism of $D^\times = (K + \epsilon K)^\times$ with $\mathrm{GL}_2(F)$:

$$\begin{aligned} \epsilon &\mapsto \begin{pmatrix} 0 & -A \\ 1 & 0 \end{pmatrix}, \\ K^\times \ni t = a + b\sqrt{A} &\mapsto \begin{pmatrix} a & bA \\ b & a \end{pmatrix}. \end{aligned}$$

Let $M = \left\{ \begin{pmatrix} y_1 & y_2 \\ 0 & 1 \end{pmatrix} \mid y_1 \in F^\times, y_2 \in F \right\}$ be the mirabolic subgroup of the standard Borel group. It carries the right invariant Haar measure $d^\times y_1\, dy_2$. As the map $K^\times \times M \to \mathrm{GL}_2(F)$, $(t, m) \mapsto t \cdot m$, is a homeomorphism ([8] Section 2.2), we may normalize the quotient measure dy on $T\backslash G$ such that $dy = d^\times y_1\, dy_2$. We identify $\phi \in \mathcal{S}(\chi, G)$ with a function in $\mathcal{S}(F^\times \times F)$,

$$\phi(y_1, y_2) := \phi \begin{pmatrix} y_1 & y_2 \\ 0 & 1 \end{pmatrix}.$$

ϕ being locally constant with compact support, there are finitely many points $(z_1, z_2) \in F^\times \times F$ and $m > 0$ such that

$$\phi(y_1, y_2) = \sum_{(z_1, z_2)} \phi(z_1, z_2) \mathbf{1}_{z_1(1+\wp^m)}(y_1) \mathbf{1}_{z_2+\wp^m}(y_2).$$

Applying this for I_ϕ and ψ,

$$I_\phi(y_1, y_2) = \sum_{(z_1, z_2)} I_\phi(z_1, z_2) \mathbf{1}_{z_1(1+\wp^m)}(y_1) \mathbf{1}_{z_2+\wp^m}(y_2),$$

$$\psi(y_1, y_2) = \sum_{(w_1, w_2)} \psi(w_1, w_2) \mathbf{1}_{w_1(1+\wp^m)}(y_1) \mathbf{1}_{w_2+\wp^m}(y_2),$$

we compute the translated local linking number

$$\begin{aligned} < \phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \psi >_x &= \int_{T\backslash G} I_\phi(y) \bar{\psi}(y \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix})\, dy \\ &= \sum_{(z_1, z_2), (w_1, w_2)} I_\phi(z_1, z_2) \bar{\psi}(w_1, w_2) \mathbf{1}_{z_2+\wp^m}(w_2) \mathbf{1}_{\frac{w_1}{z_1}(1+\wp^m)}(b) \\ &\qquad \cdot \operatorname{vol}^\times(1 + \wp^m) \operatorname{vol}(\wp^m). \end{aligned}$$

We have proved:

Theorem 4.1. *Let T be compact. For fixed x, the translated local linking number $<\phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \psi >_x$ is a locally constant function of $b \in F^\times$ with compact support.*

We give an explicit example used later on.

Example 4.2. *([9], Bsp. 4.8) Let K/F be an unramified field extension and let $\chi = 1$. Then $\phi = \chi \cdot \mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$ is well defined in $\mathcal{S}(\chi, G)$ and*

$$< \phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \phi >_x \cdot \operatorname{vol}^{-1} =$$

$$\mathbf{1}_{\mathrm{N}\setminus(1+\wp)}(x)\mathbf{1}_{\mathbf{o}_F^\times}(b) + \mathbf{1}_{1+\wp}(x)\big(\mathbf{1}_{(1-x)\mathbf{o}_F^\times}(b) + \mathbf{1}_{(1-x)^{-1}\mathbf{o}_F^\times}(b)\big)q^{-v(1-x)},$$

where $\operatorname{vol} := \operatorname{vol}_T(T) \operatorname{vol}^\times(\mathbf{o}_F^\times) \operatorname{vol}(\mathbf{o}_F)$.

4.2. **The noncompact case.** Let $K = F \oplus F$ be a split algebra. The character χ is of the form $\chi = (\chi_1, \chi_1^{-1})$ for a character χ_1 of $F^\times$. As in the proof of Proposition 3.2, $G = TNN' \cup TNwN$. Both of these open subsets are invariant under right translation by $\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$. Choose coset representatives for $T\backslash TNN'$ of the form

$$y = \begin{pmatrix} 1 & y_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_3 & 1 \end{pmatrix}$$

as well as coset representatives for $T\backslash TNwN$ of the form

$$y = \begin{pmatrix} 1 & y_1 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & 0 \\ y_4 & 1 \end{pmatrix}.$$

Any function $\psi \in \mathcal{S}(\chi, G)$ can be split into a sum $\psi = \psi_1 + \psi_2$, $\psi_i \in \mathcal{S}(\chi, G)$, with $\operatorname{supp}\psi_1 \subset TNN'$ (resp. $\operatorname{supp}\psi_2 \subset TNwN$). The function ψ_1 can be viewed as an element of $\mathcal{S}(F^2)$ in the variable (y_2, y_3). Choose the quotient measure dy on $T\backslash TNN'$ such that $dy = dy_2\, dy_3$ for fixed Haar measure dy_i on F. Proceed analogously for ψ_2. For fixed x the inner integral I_ϕ (13) is a locally constant function in y. Its support is not compact anymore, but I_ϕ is the locally constant limit of Schwartz functions. This is the reason for this case being that more elaborate than the case of a compact torus. The shape of the translated linking numbers is given by the following theorem.

Theorem 4.3. *Let T be a noncompact torus. For fixed x, the translated local linking number $< \phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \psi >_x$ is a function in $b \in F^\times$ of the form*

$$\chi_1^{-1}(b)\Big(\mathbf{1}_{\wp^n}(b)|b|(a_{+,1}v(b) + a_{+,2}) + A(b) + \mathbf{1}_{\wp^n}(b^{-1})|b|^{-1}(a_{-,1}v(b) + a_{-,2})\Big)$$

$$+ \chi_1(b)\Big(\mathbf{1}_{\wp^n}(b)|b|(c_{+,1}v(b) + c_{+,2}) + C(b) + \mathbf{1}_{\wp^n}(b^{-1})|b|^{-1}(c_{-,1}v(b) + c_{-,2})\Big),$$

with suitable constants $a_{\pm,i}, c_{\pm,i} \in \mathbb{C}$, integral $n > 0$ and functions $A, C \in \mathcal{S}(F^\times)$.

Sketch of proof of Theorem 4.3. This is done by brute force computations in [9] Chapter 8. We will outline the reduction to $\wp$-adic integration here. We choose the functions ϕ, ψ locally as simple as possible: $z \in \operatorname{supp}\phi$ belongs to TNN' or $TNwN$. We restrict to $z \in TNN'$, the other case is done similarly. There is a representative

$$\begin{pmatrix} 1+z_2z_3 & z_2 \\ z_3 & 1 \end{pmatrix}$$

of z modulo T and an open set

$$U_z = \begin{pmatrix} 1+z_2z_3 & z_2 \\ z_3 & 1 \end{pmatrix} + \begin{pmatrix} \wp^m & \wp^m \\ \wp^m & \wp^m \end{pmatrix}$$

such that $\phi|_{U_z} = \phi(z)$. Choosing m that large that U_z is fundamental, ϕ locally has the shape $\phi_z := \chi \cdot \mathbf{1}_{U_z}$ up to some multiplicative constant. For the exterior function ψ proceed similarly. It is enough to determine the behavior of the translated local linking numbers for functions of this type, i.e.

$$\int_{T\backslash G} \int_T \phi_z(t^{-1}\gamma(x)ty)\ dt\ \bar{\psi}_{\tilde{z}}(y \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix})\ dy.$$

According to whether z_2 or z_3 is zero or not, and $\operatorname{supp}\psi \subset TNN'$ or $\operatorname{supp}\psi \subset TNwN$, there are $2^3 = 8$ types of integrals to be done ([9] Chapters 5.2 and 8). For later use we include an explicit example.

Example 4.4. *([9], Bsp. 5.2) Let T be noncompact. Let $\chi = (\chi_1, \chi_1)$, where χ_1 is unramified and quadratic. Then $\phi = \chi \cdot \mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$ is well-defined in $\mathcal{S}(\chi, G)$. The translated local linking number $< \phi, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \phi >_x$ is given by*

$$\begin{aligned}
&\chi_1(1-x)\chi_1(b)\operatorname{vol}^\times(\mathbf{o}_F^\times)\operatorname{vol}(\mathbf{o}_F)^2 \cdot \\
&\Big[\mathbf{1}_{F^\times\backslash(1+\wp)}(x)\Big(\mathbf{1}_{\mathbf{o}_F^\times}(b)\big(|v(x)|+1\big)(1+q^{-1}) + \mathbf{1}_\wp(b)|b|\big(4v(b)+2|v(x)|\big) \\
&\qquad\qquad + \mathbf{1}_\wp(b^{-1})|b^{-1}|\big(-4v(b)+2|v(x)|\big)\Big) \\
&\quad + \quad \mathbf{1}_{1+\wp}(x)\Big(\mathbf{1}_{\wp^{v(1-x)+1}}(b)|b|\big(4v(b)-4v(1-x)\big) \\
&\qquad\qquad + \mathbf{1}_{v(1-x)\mathbf{o}_F^\times}(b)|b| + \mathbf{1}_{v(1-x)\mathbf{o}_F^\times}(b^{-1})|b^{-1}| \\
&\qquad\qquad + \mathbf{1}_{\wp^{v(1-x)+1}}(b^{-1})|b^{-1}|\big(-4v(b)-4v(1-x)\big)\Big)\Big].
\end{aligned}$$

5. A geometric Hecke operator

We construct operators on the local linking numbers that realize the asymptotics ($b \to 0$) of the Hecke operators on Whittaker products. The asymptotics of the second is as follows.

Proposition 5.1. *The Whittaker products $W(b\xi, b\eta)$ have the following behavior for $b \to 0$ and fixed $\xi = \frac{x}{x-1}$, $\eta = 1-\xi$.*
(a) In case of a compact Torus T and χ not factorizing via the norm,

$$W(b\xi, b\eta) \;=\; 0.$$

In case of a compact Torus T and $\chi = \chi_1 \circ \mathrm{N}$,

$$W(b\xi, b\eta) = |b||\xi\eta|^{\frac{1}{2}}\chi_1(b\eta)\left(c_1 + c_2\omega(b\xi)\right)\left(c_3\mathbf{1}_{\wp^m \cap (1-x)\mathrm{N}}(b) + c_4\mathbf{1}_{\wp^m\cap(1-x)z\mathrm{N}}(b)\right),$$

where $z \in F^\times \backslash \mathrm{N}$.
(b) In case of a noncompact Torus T,

$$W(b\xi, b\eta) = \begin{cases} |b||\xi\eta|^{\frac{1}{2}}\left(c_1\chi_1(b\eta) + c_2\chi_1^{-1}(b\eta)\right)\left(c_3 v(b\xi) + c_4\right)), & \text{if } \chi_1^2 \neq 1 \\ |b||\xi\eta|^{\frac{1}{2}}\chi_1(b\eta)\left(c_1 v(b\eta) + c_2\right)\left(c_3 v(b\xi) + c_4\right), & \text{if } \chi_1^2 = 1 \end{cases}.$$

Here, $c_i \in \mathbb{C}$, $i = 1, \dots, 4$.

Proof of Proposition 5.1. For $b \to 0$ both arguments $b\xi$ and $b\eta$ tend to zero. The stated behaviors are collected from Propositions 2.9 and 2.10. □

Notice that the translation by b of the local linking numbers underlies this asymptotics (Theorems 4.1 and 4.3), but it does not realize the leading terms in case χ is quadratic. In case of a noncompact torus T, the leading term is $v(b)^2$, while translation only produces $v(b)$. In case of a compact torus, the translated linking numbers have compact support, while the Hecke operator on Whittaker products has not. In the following, we make the additional "completely unramified" assumption which is satisfied at almost all places.

Hypothesis 5.2. *D is a split algebra. K/F is an unramified extension (split or nonsplit) contained in D. The character χ is unramified.*

For a noncompact torus T the translated local linking numbers (Theorem 4.3) split into sums of the form

$$<\phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\psi>_x = <\phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\psi>_x^+ + <\phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\psi>_x^-,$$

where

$$<\phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\psi>_x^{\pm} := \chi_1^{\pm 1}(\beta)\cdot \left(\mathbf{1}_{\wp^n}|\beta|(c_{\pm,1}v(\beta) + c_{\pm,2}) + C_{\pm}(\beta) + \mathbf{1}_{\wp^n}(\beta^{-1})|\beta|^{-1}(d_{\pm,1}v(\beta) + d_{\pm,2})\right) \tag{14}$$

are the summands belonging to $\chi_1^{\pm 1}$ respectively. In here, the constants $c_{\pm,i}, d_{\pm,i}$, and $C_\pm \in \mathcal{S}(F^\times)$ as well as $n > 0$ depend on ϕ, ψ and x. If χ_1 is a quadratic character, these two summands coincide. To give an operator fitting all cases, define in case of a compact torus

$$< \phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \psi >_x^{\pm} := < \phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \psi >_x .$$

For $v(b) \geq 0$ define the operator $\mathbf{S}_b$ to be

$$\mathbf{S}_b := \frac{1}{4} \left(\mathbf{S}_b^+ + \mathbf{S}_b^- \right), \tag{15}$$

where

$$\begin{aligned} \mathbf{S}_b^{\pm} < \phi, \psi >_x \quad := \quad & \sum_{s=0,1} \sum_{i=0}^{v(b)} \frac{\chi_1^{\mp 1}(\pi)^{i(-1)^s} \omega(b(1-x))^{i+s}}{|\pi^{v(b)-i}|} \\ & \cdot < \phi, \begin{pmatrix} \pi^{(-1)^s(v(b)-i)} & 0 \\ 0 & 1 \end{pmatrix} \psi >_x^{\pm} . \end{aligned}$$

This “Hecke operator” is not unique. For example, the summand for $s = 0$ has the same properties as $\mathbf{S}_b$ itself. The crucial point is that an averaging sum occurs. The operator $\mathbf{S}_b$ is chosen such that this sum includes negative exponents $-v(b) + i$ as well. This kind of symmetry will make the results on the local Gross-Zagier formula look smoothly (Section 6.2).

Proposition 5.3. *Let T be a compact torus. Then the operator $\mathbf{S}_b$ reduces to*

$$\mathbf{S}_b < \phi, \psi >_x = \frac{1}{2} \sum_{s=0,1} \sum_{i=0}^{v(b)} \frac{\omega(b(1-x))^{i+s}}{|\pi^{v(b)-i}|} < \phi, \begin{pmatrix} \pi^{(-1)^s(v(b)-i)} & 0 \\ 0 & 1 \end{pmatrix} \psi >_x .$$

Let $x \in c\mathrm{N}$ be fixed. For $\phi, \psi \in \mathcal{S}(\chi, G)$ there are constants $c_1, c_2 \in \mathbb{C}$ and $n \in \mathbb{N}$ such that for $v(b) \geq n$

$$\mathbf{S}_b < \phi, \psi >_x = c_1 \mathbf{1}_{\wp^n \cap (1-x)\mathrm{N}}(b) + c_2 \mathbf{1}_{\wp^n \cap (1-x)z\mathrm{N}}(b).$$

Proposition 5.4. *Let T be a noncompact torus. The operators $\mathbf{S}_b^{\pm}$ reduce to*

$$\mathbf{S}_b^{\pm} < \phi, \psi >_x = \sum_{s=0,1} \sum_{i=0}^{v(b)} \frac{\chi_1^{\mp 1}(\pi)^{i(-1)^s}}{|\pi^{v(b)-i}|} < \phi, \begin{pmatrix} \pi^{(-1)^s(v(b)-i)} & 0 \\ 0 & 1 \end{pmatrix} \psi >_x^{\pm} .$$

Let $x \in F^\times$ be fixed. For $\phi, \psi \in \mathcal{S}(\chi, G)$ there are constants $c_0, \ldots, c_3 \in \mathbb{C}$ and $n \in \mathbb{N}$ such that for $v(b) \geq n$

$$\mathbf{S}_b < \phi, \psi >_x = \begin{cases} \chi_1^{-1}(b)\big(c_3 v(b) + c_2\big) + \chi_1(b)\big(c_1 v(b) + c_0\big), & \text{if } \chi_1^2 \neq 1 \\ \chi_1(b)\big(c_2 v(b)^2 + c_1 v(b) + c_0\big), & \text{if } \chi_1^2 = 1 \end{cases} .$$

Theorem 5.5. *For fixed x, the local linking numbers $|b|^{-1}|\xi\eta|^{\frac{1}{2}}\mathbf{S}_b < \phi, \psi >_x$ and the Whittaker products $\mathbf{T}_b W(\xi, \eta)$ have the same asymptotics in b.*

Proof of Theorem 5.5. Recall that $\mathbf{T}_b W(\xi, \eta) = |b|^{-2} W(b\xi, b\eta)$. In case T compact, combine Proposition 5.3 and Proposition 5.1 (a) for $\chi = 1$. In case T noncompact, combine Proposition 5.4 and Proposition 5.1 (b). $\square$

Proof of Proposition 5.3. For T compact Assumtion 5.2 induces $\chi = 1$ by Corollary 2.6. By Theorem 4.1, the translated linking number can be written as

$$< \phi, \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \psi >_x = \sum_i d_{a_i} |a_i|^{\operatorname{sign} v(a_i)} \mathbf{1}_{a_i(1+\wp^m)}(\beta),$$

for finitely many $a_i \in F^\times$, $d_{a_i} \in \mathbb{C}$, and some $m > 0$, where the sets $a_i(1+\wp^m)$ are pairwise disjoint. We may assume that in this sum all π^l, $-\max_i |v(a_i)| \leq l \leq \max_i |v(a_i)|$, occur. Let $n := \max_i |v(a_i)| + 1$. Then, for $v(b) \geq n$,

$$\begin{aligned}
\mathbf{S}_b < \phi, \psi >_x &= \frac{1}{2} \sum_{i=0}^{v(b)} \Bigg(\omega(b(1-x))^i \sum_{l=-n+1}^{n-1} \frac{d_{\pi^l} |\pi^l|^{\operatorname{sign}(l)}}{|\pi^{v(b)-i}|} \mathbf{1}_{\pi^l(1+\wp^m)}(\pi^{v(b)-i}) \\
&\quad + \omega(b(1-x))^{i+1} \sum_{l=-n+1}^{n-1} \frac{d_{\pi^l} |\pi^l|^{\operatorname{sign}(l)}}{|\pi^{v(b)-i}|} \mathbf{1}_{\pi^l(1+\wp^m)}(\pi^{i-v(b)}) \Bigg) \\
&= \frac{1}{2} \sum_{l=0}^{n-1} \omega(b(1-x))^{v(b)+l} d_{\pi^l} + \frac{1}{2} \sum_{l=-n+1}^{0} \omega(b(1-x))^{v(b)+l+1} d_{\pi^l} \\
&= c_1 \mathbf{1}_{\wp^n \cap (1-x)\,\mathrm{N}}(b) + c_2 \mathbf{1}_{\wp^n \cap (1-x) z\,\mathrm{N}}(b),
\end{aligned}$$

where $c_1 := \frac{1}{2} \sum_{l=0}^{n-1} (d_{\pi^l} + d_{\pi^{-l}})$ and $c_2 := \frac{1}{2} \sum_{l=0}^{n-1} (-1)^l (d_{\pi^l} - d_{\pi^{-l}})$. Notice, that for $b(1-x) \in z\,\mathrm{N}$ one has $\omega(b(1-x))^{v(b)} = (-1)^{v(b)} = -\omega(1-x)$. $\square$

Proof of Proposition 5.4. T is noncompact, so $\omega = 1$. First we prove this asymptotics for the part $\mathbf{T}_b^-$ of $\mathbf{S}_b$ belonging to $\mathbf{S}_b^-$ and $s = 0$,

$$\mathbf{T}_b^- < \phi, \psi >_x := \sum_{i=0}^{v(b)} \frac{\chi_1(\pi)^i}{|\pi^{v(b)-i}|} < \phi, \begin{pmatrix} \pi^{v(b)-i} & 0 \\ 0 & 1 \end{pmatrix} \psi >_x^- .$$

Let $n > 0$ be the integer of (14). Let $v(b) \geq n$. In the formula for $\mathbf{T}_b^-$, we distinguish the summands whether $v(b) - i < n$ or not. If $v(b) - i < n$, then

$$< \phi, \begin{pmatrix} \pi^{v(b)-i} & 0 \\ 0 & 1 \end{pmatrix} \psi >_x^- = \chi_1^{-1}(\pi^{v(b)-i}) C_-(\pi^{v(b)-i}).$$

The function $\tilde{C}_-$ defined by

$$\tilde{C}_-(\beta) := \frac{\chi_1^{-2}(\beta)}{|\beta|} C_-(\beta)$$

belongs to $\mathcal{S}(F^\times)$. The part of $\mathbf{T}_b^-$ made up by summands satisfying $v(b)-i<n$ is now simplified to

$$\begin{aligned}\sum_{i=v(b)-n+1}^{v(b)} &\frac{\chi_1(\pi)^i}{|\pi^{v(b)-i}|} < \phi, \begin{pmatrix}\pi^{v(b)-i} & 0\\ 0 & 1\end{pmatrix}\psi >_x^- \\ &= \sum_{i=v(b)-n+1}^{v(b)} \chi_1(b)\tilde{C}_-(\pi^{v(b)-i}) \;=\; \chi_1(b)\sum_{l=0}^{n-1}\tilde{C}_-(\pi^l).\end{aligned}$$

In here, the last sum is independent of b. Thus, this part of $\mathbf{T}_b^-$ satisfies the claim. In the remaining part

$$\mathbf{T}(i\le v(b)-n) \;:=\; \sum_{i=0}^{v(b)-n} \frac{\chi_1(\pi)^i}{|\pi^{v(b)-i}|} < \phi, \begin{pmatrix}\pi^{v(b)-i} & 0\\ 0 & 1\end{pmatrix}\psi >_x^-$$

all the translated local linking numbers occuring can be written as

$$< \phi, \begin{pmatrix}\pi^{v(b)-i} & 0\\ 0 & 1\end{pmatrix}\psi >_x^- = \chi_1^{-1}(\pi^{v(b)-i})|\pi^{v(b)-i}|\left(c_{-,1}(v(b)-i)+c_{-,2}\right).$$

So

$$\mathbf{T}(i\le v(b)-n) = \chi_1^{-1}(b)\sum_{i=0}^{v(b)-n}\chi_1(\pi)^{2i}\left(c_{-,1}(v(b)-i)+c_{-,2}\right).$$

In case $\chi_1^2=1$, we have

$$\mathbf{T}(i\le v(b)-n) = \chi_1(b)(v(b)-n+1)\left(c_{-,2}+\frac{1}{2}c_{-,1}(v(b)+n)\right),$$

which owns the claimed asymptotics. If $\chi_1^2\ne 1$, enlarge n such that $\chi_1^n=1$. The remaining part of $\mathbf{T}_b^-$ then is

$$\begin{aligned}\mathbf{T}(i\le v(b)-n) = (c_{-,1}v(b)+c_{-,2})&\frac{\chi_1(b\pi)-\chi_1^{-1}(b\pi)}{\chi_1(\pi)-\chi_1^{-1}(\pi)} \\ &- c_{-,1}\frac{\chi_1(b\pi)(v(b)-n+1)}{\chi_1(\pi)-\chi_1^{-1}(\pi)} + c_{-,1}\frac{\chi_1(b\pi^2)-1}{(\chi_1(\pi)-\chi_1^{-1}(\pi))^2}.\end{aligned}$$

Thus, the claim is satisfied in case $\chi_1^2\ne 1$. The other parts of $\mathbf{S}_b$ satisfy the claimed asymptotics as well: If $\mathbf{T}_b^+$ denotes the part of $\mathbf{S}_b$ belonging to $\mathbf{S}_b^+$ and $s=0$, then the statement for $\mathbf{T}_b^+$ follows from the proof for $\mathbf{T}_b^-$ replacing there χ_1^{-1} by χ_1, C_- by C_+, and $c_{-,i}$ by $c_{+,i}$, where the constants are given by (14). For $s=1$ notice that

$$\chi_1(\pi)^{i(-1)^s}\chi_1^{-1}(\pi^{(-1)^s(v(b)-i)}) = \chi_1(b)\chi_1(\pi)^{-2i}.$$

So the claim follows from the proof for $s=0$ if there we substitute χ_1 by χ_1^{-1} as well as $c_{\pm,i}$ by $d_{\pm,i}$ of (14). $\square$

6. Local Gross-Zagier formula

We report on Zhang's local Gross-Zagier formulae for GL_2 [13] using our notations in order to compare them directly with the results given by the operator $\mathbf{S}_b$. We include short proofs of S. Zhang's results.

6.1. **S. Zhang's local Gross-Zagier formula.** The local Gross-Zagier formula compares the Whittaker products of local newforms with a local linking number belonging to a very special function ϕ ([13] Chapter 4.1),

$$\phi \;=\; \chi \cdot \mathbf{1}_{R^\times},$$

where $R^\times$ is the unit group of a carefully chosen order R in D. Almost everywhere, especially under Hypothesis 5.2, $R^\times = \mathrm{GL}_2(\mathbf{o}_F)$ and the function ϕ is well-defined. The specially chosen local linking number then is

$$< \tilde{\mathbf{T}}_b\phi, \phi >_x,$$

where the geometric Hecke operator $\tilde{\mathbf{T}}_b$ is defined as follows ([13] 4.1.22 et sqq.). Let

$$H(b) := \{g \in M_2(\mathbf{o}_F) \mid v(\det g) = v(b)\}.$$

Then

$$\tilde{\mathbf{T}}_b\phi(g) := \int_{H(b)} \phi(hg)\; dh.$$

This operator is well-defined on $\phi = \chi \cdot \mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$, but not generally on $\mathcal{S}(\chi, G)$. In our construction of the universal operator $\mathbf{S}_b$ we followed the idea that $\tilde{\mathbf{T}}_b$ reflects summation over translates by coset representatives, as

$$H(b) = \bigcup \begin{pmatrix} y_1 & 0 \\ 0 & y_3 \end{pmatrix} \begin{pmatrix} 1 & y_2 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathbf{o}_F),$$

where the union is over representatives $(y_1, y_3) \in \mathbf{o}_F \times \mathbf{o}_F$ with $v(y_1y_3) = v(b)$ and $y_2 \in \wp^{-v(y_1)} \backslash \mathbf{o}_F$.

Lemma 6.1. *([13] Lemma 4.2.2) Let K/F be a field extension and assume Hypothesis 5.2. Let $\phi = \chi \cdot \mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$. Then*

$$< \tilde{\mathbf{T}}_b\phi, \phi >_x = \; \mathrm{vol}(\mathrm{GL}_2(\mathbf{o}_F))^2 \,\mathrm{vol}_T(T) \mathbf{1}_{\mathrm{N}}(x) \mathbf{1}_{\frac{1-x}{x}(\mathbf{o}_F \cap \mathrm{N})}(b) \mathbf{1}_{(1-x)(\mathbf{o}_F \cap \mathrm{N})}(b).$$

Lemma 6.2. *([13] Lemma 4.2.3) Let K/F be split, let $\chi = (\chi_1, \chi_1^{-1})$ be an unramified character, and let $\phi = \chi \cdot \mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$. In case $\chi_1^2 \neq 1$,*

$$< \tilde{\mathbf{T}}_b\phi, \phi >_x = \frac{\chi_1(b(1-x)^{-1}\pi) - \chi_1^{-1}(b(1-x)^{-1}\pi)}{\chi_1(\pi) - \chi_1^{-1}(\pi)} \,\mathrm{vol}(\mathrm{GL}_2(\mathbf{o}_F))^2 \,\mathrm{vol}^\times(\mathbf{o}_F^\times)$$
$$\cdot\, \mathbf{1}_{\frac{1-x}{x}\mathbf{o}_F \cap (1-x)\mathbf{o}_F}(b) \mathbf{1}_{F^\times}(x) \left(v(b) + v(\frac{x}{1-x}) + 1 \right).$$

In case $\chi_1^2 = 1$,

$$< \tilde{\mathbf{T}}_b\phi, \phi >_x = \chi_1(b(1-x)) \operatorname{vol}(\mathrm{GL}_2(\mathbf{o}_F))^2 \operatorname{vol}^\times(\mathbf{o}_F^\times)\mathbf{1}_{\frac{1-x}{x}\mathbf{o}_F\cap(1-x)\mathbf{o}_F}(b) \cdot \mathbf{1}_{F^\times}(x)\big(v(b)-v(1-x)+1\big)\big(v(b)+v(\frac{x}{1-x})+1\big).$$

For the proofs of Lemma 6.1 and 6.2 we follow a hint by Uwe Weselmann. Write

$$\phi(x) \;=\; \sum_{\tau\in T(F)/T(\mathbf{o}_F)} \chi(\tau)\mathbf{1}_{\tau\,\mathrm{GL}_2(\mathbf{o}_F)}(x).$$

For the Hecke operator we find

$$\tilde{\mathbf{T}}_b\mathbf{1}_{\tau\,\mathrm{GL}_2(\mathbf{o}_F)}(x) \;=\; \operatorname{vol}(\mathrm{GL}_2(\mathbf{o}_F))\mathbf{1}_{\tau b^{-1}H(b)}(x),$$

as $b^{-1}H(b) = \{h \in \mathrm{GL}_2(F) \mid h^{-1} \in H(b)\}$. As the Hecke operator is invariant under right translations, $\tilde{\mathbf{T}}_b\phi(xy) = \tilde{\mathbf{T}}_b\phi(x)$ for $y \in \mathrm{GL}_2(\mathbf{o}_F)$, we get

$$(16) \qquad < \tilde{\mathbf{T}}_b\phi, \phi >_x = \operatorname{vol}(\mathrm{GL}_2(\mathbf{o}_F))^2 \sum_\tau \chi(\tau)\int_T \mathbf{1}_{\tau b^{-1}H(b)}(t^{-1}\gamma(x)t)\;dt.$$

This formula is evaluated in the different cases for K/F.

Proof of Lemma 6.1. Let $K = F(\sqrt{A})$, where $v(A) = 0$. Choose a trace zero $\gamma(x) = \sqrt{A} + \epsilon(\gamma_1 + \gamma_2\sqrt{A})$, where $\mathrm{N}(\gamma_1 + \gamma_2\sqrt{A}) = x$. The conditions for the integrands of (16) not to vanish are

$$\tau^{-1}b\sqrt{A} \in \mathbf{o}_K$$
$$\tau^{-1}b\bar{t}^{-1}t(\gamma_1+\gamma_2\sqrt{A}) \in \mathbf{o}_K$$
$$\det(t^{-1}\gamma(x)t) = A(x-1) \in b^{-1}\,\mathrm{N}(\tau)\mathbf{o}_F^\times.$$

They are equivalent to $|\mathrm{N}(\tau)| = |b(1-x)|$ and $|b| \le \min\{|\frac{1-x}{x}|, |1-x|\}$. There is at most one coset $\tau \in T(F)/T(\mathbf{o}_F)$ satisfying this, and this coset exists only if $b \in (1-x)\,\mathrm{N}$. Thus,

$$< \tilde{\mathbf{T}}_b\phi, \phi >_x \;=\; \operatorname{vol}(\mathrm{GL}_2(\mathbf{o}_F))^2 \operatorname{vol}_T(T) \cdot \left(\mathbf{1}_{\mathrm{N}\setminus(1+\wp)}(x)\mathbf{1}_{\mathbf{o}_F\cap\mathrm{N}}(b) + \mathbf{1}_{1+\wp}(x)\mathbf{1}_{(1-x)(\mathbf{o}_F\cap\mathrm{N})}(b)\right),$$

which equals the claimed result. □

Proof of Lemma 6.2. Choose $\gamma(x) = \begin{pmatrix} -1 & x \\ -1 & 1 \end{pmatrix}$ of trace zero, and set $\tau = (\tau_1, \tau_2) \in K^\times/\mathbf{o}_K^\times$ as well as $t = (a, 1) \in T$. The conditions for an integrand of (16) not to vanish are

$$(-\tau_1^{-1}b, \tau_2^{-1}b) \in \mathbf{o}_K,$$
$$(-\tau_1^{-1}a^{-1}bx, \tau_2^{-1}ab) \in \mathbf{o}_K,$$
$$\det(t^{-1}\gamma(x)t) = x-1 \in \mathrm{N}(\tau)b^{-1}\mathbf{o}_K^\times.$$

So only if $v(\tau_2) = -v(\tau_1) + v(b) + v(1-x)$ satisfies $v(1-x) \leq v(\tau_2) \leq v(b)$, the integral does not vanish. Then the scope of integration is given by $-v(b) + v(\tau_2) \leq v(a) \leq v(\tau_2) + v(x) - v(1-x)$ and the integral equals

$$\mathrm{vol}^\times(\mathbf{o}_F^\times)\big(v(b) + v(x) - v(1-x) + 1\big)\mathbf{1}_{\mathbf{o}_F \cap \wp^{v(1-x)-v(x)}}(b).$$

Evaluating $\chi(\tau)$ we get $\chi(\tau) = \chi_1(b(1-x))\chi_1^{-2}(\tau_2)$, as χ is unramified. Summing up the terms of (16) yields the lemma. $\square$

The other constituents of the local Gross-Zagier formulae are the Whittaker products of newforms for both the Theta series $\Pi(\chi)$ and the Eisenstein series $\Pi(1,\omega)$ at $s = \frac{1}{2}$. By Hypothesis 5.2, the Theta series equals $\Pi(\chi_1, \chi_1^{-1})$ if K/F splits, and it equals $\Pi(1,\omega)$ if K/F is a field extension. Thus, all occuring representations are principal series and the newforms read in the Kirillov model are given by (8). In case of a field extension we get

$$W_{\theta,new}(a) \;=\; W_{E,new}(a) \;=\; \mathrm{vol}(\mathbf{o}_F)\,\mathrm{vol}^\times(\mathbf{o}_F^\times)\cdot|a|^{\frac{1}{2}}\mathbf{1}_{\mathbf{o}_F\cap \mathrm{N}}(a)\,.$$

In case K/F splits we get

$$W_{\theta,new}(a) \;=\; \mathrm{vol}(\mathbf{o}_F)\,\mathrm{vol}^\times(\mathbf{o}_F^\times)\cdot|a|^{\frac{1}{2}}\mathbf{1}_{\mathbf{o}_F}(a)\begin{cases}\frac{\chi_1(a\pi)-\chi_1^{-1}(a\pi)}{\chi_1(\pi)-\chi_1^{-1}(\pi)}, & \text{if } \chi_1^2 \neq 1\\ \chi_1(a)(v(a)+1), & \text{if } \chi_1^2 = 1\end{cases},$$

while

$$W_{E,new}(a) \;=\; \mathrm{vol}(\mathbf{o}_F)\,\mathrm{vol}^\times(\mathbf{o}_F^\times)\cdot|a|^{\frac{1}{2}}\mathbf{1}_{\mathbf{o}_F}(a)(v(a)+1)\,.$$

Summing up, we get the following Lemma. Recall $\xi = \frac{x}{x-1}$ and $\eta = 1 - \xi$.

Lemma 6.3. ([13] *Lemma 3.4.1) Assume Hypothesis 5.2. Then the Whittaker products for the newforms of Theta series and Eisenstein series have the following form up to the factor* $\mathrm{vol}(\mathbf{o}_F)^2\,\mathrm{vol}^\times(\mathbf{o}_F^\times)^2$. *If K/F is a field extension, then*

$$\begin{aligned} W_{\theta,new}(b\eta)W_{E,new}(b\xi) &= |\xi\eta|^{\frac{1}{2}}|b|\mathbf{1}_{\mathbf{o}_F}(b\xi)\mathbf{1}_{\mathbf{o}_F}(b\eta)\\ &= |\xi\eta|^{\frac{1}{2}}|b|\mathbf{1}_{\frac{1-x}{x}(\mathbf{o}_F\cap\mathrm{N})}(b)\mathbf{1}_{(1-x)(\mathbf{o}_F\cap\mathrm{N})}(b). \end{aligned}$$

If K/F splits and χ is quadratic, then

$$\begin{aligned} &W_{\theta,new}(b\eta)W_{E,new}(b\xi)\\ &\quad= |\xi\eta|^{\frac{1}{2}}|b|\mathbf{1}_{\mathbf{o}_F}(b\xi)\mathbf{1}_{\mathbf{o}_F}(b\eta)\chi_1(b\eta)\,(v(b\xi)+1)\,(v(b\eta)+1)\\ &\quad= |\xi\eta|^{\frac{1}{2}}|b|\mathbf{1}_{\frac{1-x}{x}\mathbf{o}_F\cap(1-x)\mathbf{o}_F}(b)\chi_1(b(1-x))\big(v(b)+v(\frac{x}{1-x})+1\big)\cdot\\ &\qquad\qquad\cdot\big(v(b)-v(1-x)+1\big). \end{aligned}$$

If K/F splits and χ is not quadratic, then

$$\begin{aligned} & W_{\theta,new}(b\eta)W_{E,new}(b\xi) \\ & \quad = |\xi\eta|^{\frac{1}{2}}|b|\mathbf{1}_{\mathbf{o}_F}(b\xi)\mathbf{1}_{\mathbf{o}_F}(b\eta)\left(v(b\xi)+1\right)\frac{\chi_1(b\eta\pi)-\chi_1^{-1}(b\eta\pi)}{\chi_1(\pi)-\chi_1^{-1}(\pi)} \\ & \quad = |\xi\eta|^{\frac{1}{2}}|b|\mathbf{1}_{\frac{1-x}{x}\mathbf{o}_F\cap(1-x)\mathbf{o}_F}(b)\big(v(b)+v(\frac{x}{1-x})+1\big)\cdot \\ & \qquad \cdot\frac{\chi_1(b(1-x)^{-1}\pi)-\chi_1^{-1}(b(1-x)^{-1}\pi)}{\chi_1(\pi)-\chi_1^{-1}(\pi)}. \end{aligned}$$

Comparing Lemma 6.1 resp. 6.2 with Lemma 6.3 we get S. Zhang's local Gross-Zagier formula:

Theorem 6.4. *([13] Lemma 4.3.1) Assume Hypothesis 5.2. Let $W_{\theta,new}$ resp. $W_{E,new}$ be the newform for the Theta series resp. Eisenstein series. Let $\phi = \chi\cdot\mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$. Then up to a factor of volumes,*

$$W_{\theta,new}(b\eta)W_{E,new}(b\xi) \;=\; |\xi\eta|^{\frac{1}{2}}|b| < \tilde{\mathbf{T}}_b\phi,\phi >_{x=\frac{\xi}{\xi-1}} .$$

6.2. **Reformulation of local Gross-Zagier.** We re-prove S. Zhang's local Gross-Zagier formula in terms of $\mathbf{S}_b$:

Theorem 6.5. *Assume Hypothesis 5.2 and assume $\chi_1^2 = 1$ in case K/F splits. Let $W_{\theta,new}$ resp. $W_{E,new}$ be the newform for the Theta series resp. Eisenstein series. Let $\phi = \chi\cdot\mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$. Then up to a factor of volumes,*

$$W_{\theta,new}(b\eta)W_{E,new}(b\xi) \;=\; |\xi\eta|^{\frac{1}{2}}|b|\mathbf{S}_b < \phi,\phi >_x + O(v(b)),$$

where in case K/F a field extension the term of $O(v(b))$ is actually zero, while in case K/F split the term of $O(v(b))$ can be given precisely by collecting terms in the proof of Example 4.4.

Proof of Theorem 6.5. Compare the Whittaker products for newforms given in Lemma 6.3 with the action of the operator $\mathbf{S}_b$ on the local linking number belonging to ϕ, given by Lemma 6.6 resp. 6.7 below. □

Lemma 6.6. *Let K/F be a field extension. Assume Hypothesis 5.2. Let $\phi = \chi\cdot\mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$. Then up the factor $\mathrm{vol}_T(T)\,\mathrm{vol}^\times(\mathbf{o}_F^\times)\,\mathrm{vol}(\mathbf{o}_F)$,*

$$\mathbf{S}_b < \phi,\phi >_x = \mathbf{1}_{\mathrm{N}}(x)\mathbf{1}_{\frac{1-x}{x}(\mathbf{o}_F\cap\mathrm{N})}(b)\mathbf{1}_{(1-x)(\mathbf{o}_F\cap\mathrm{N})}(b).$$

Proof of Lemma 6.6. The translated local linking number is that of Example 4.2. We compute the action of $\mathbf{S}_b$ given by Proposition 5.3. If $x \in \mathrm{N}\setminus(1+\wp)$, then up to the factor $\mathrm{vol}_T(T)\,\mathrm{vol}^\times(\mathbf{o}_F^\times)\,\mathrm{vol}(\mathbf{o}_F)$,

$$\mathbf{S}_b < \phi,\phi >_x = \frac{1}{2}\left(\omega(b(1-x))^{v(b)} + \omega(b(1-x))^{v(b)+1}\right) = \mathbf{1}_{\mathrm{N}}(b).$$

If $x \in 1+\wp$, then again up to the factor of volumes

$$\begin{aligned} \mathbf{S}_b < \phi,\phi >_x \; &= \; \frac{1}{2}\mathbf{1}_{\wp^{v(1-x)}}(b)\omega(b(1-x))^{v(b)-v(1-x)}\left(1+\omega(b(1-x))\right) \\ &= \mathbf{1}_{\wp^{v(1-x)}\cap(1-x)\,\mathrm{N}}(b). \end{aligned}$$

□

In case K/F we restrict ourselves to the case $\chi_1^2 = 1$.

Lemma 6.7. *Let K/F be split and assume Hypothesis 5.2 as well as $\chi_1^2 = 1$. Let $\phi = \chi \cdot \mathbf{1}_{\mathrm{GL}_2(\mathbf{o}_F)}$. Then up the factor* $\mathrm{vol}^\times(\mathbf{o}_F^\times)\,\mathrm{vol}(\mathbf{o}_F)^2$,

$$\begin{aligned}\mathbf{S}_b < \phi, \phi >_x = \ & \chi_1(b(1-x))\cdot \\ & \Big[\mathbf{1}_{F^\times\backslash(1+\wp)}(x)\Big(2v(b)^2 + 2(|v(x)|+1)v(b) + (1+q^{-1})(|v(x)|+1)\Big) \\ & +\mathbf{1}_{1+\wp}(x)\mathbf{1}_{\wp^{v(1-x)}}(b)\Big(2\big(v(b)-v(1-x)+1\big)\big(v(b)-v(1-x)\big)+1\Big)\Big].\end{aligned}$$

Proof of Lemma 6.7. The operator $\mathbf{S}_b$ is given by Proposition 5.4. The translated local linking number is given by Example 4.4. As χ_1 is quadratic, $\mathbf{S}_b = \frac{1}{2}\mathbf{S}_b^+$. For $x \in 1+\wp$ we compute

$$\begin{aligned}&\mathbf{S}_b < \phi, \phi >_x \\ &= \chi_1(b(1-x))\mathbf{1}_{\wp^{v(1-x)}}(b)\left(1 + \sum_{i=0}^{v(b)-v(1-x)-1} 4(v(b)-i-v(1-x))\right) \\ &= \chi_1(b(1-x))\mathbf{1}_{\wp^{v(1-x)}}(b)\Big(2(v(b)-v(1-x)+1)(v(b)-v(1-x))+1\Big),\end{aligned}$$

while for $x \in F^\times\backslash(1+\wp)$,

$$\begin{aligned}&\mathbf{S}_b < \phi, \phi >_x \\ &= \chi_1(b(1-x))\left[(|v(x)|+1)(1+q^{-1}) + \sum_{i=0}^{v(b)-1}\big(4(v(b)-i)+2|v(x)|\big)\right] \\ &= \chi_1(b(1-x))\Big(2v(b)^2 + (1+|v(x)|)(2v(b)+1+q^{-1})\Big).\end{aligned}$$

□

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