A Chebotarev Density Theorem for Function Fields

Armin Holschbach

Let $f : Y \to X$ be a finite branched Galois cover of normal varieties over a field k, and let $G = \mathrm{Gal}(X/Y)$ denote its Galois group.

The Serre-Chebotarev density theorem considers the case where k is a finite field. It defines a Dirichlet density on the set of closed points of X and describes the asymptotic decomposition behavior of these points in the cover $Y \to X$ ([4, Theorem 7]).

Instead of looking at closed points, we will consider points of codimension one on X and describe "how many" of those have a given decomposition behavior:

To any codimension 1 point $x \in X$ (or the corresponding Weil prime divisor), we associate a *decomposition type* by taking the conjugacy class of the decomposition group of any point y on Y mapping to x. This notion does not depend on the choice of the point y. If the decomposition type of x is trivial, we say x *splits completely* in Y. In the following we restrict ourselves to Weil prime divisors that stay prime after finite base extensions, i.e. geometrically integral divisors.

1. Density Results for Divisors

Assume k is perfect and X, Y are projective and geometrically integral over k. Moreover, assume $d := \dim X \geq 2$ and $\mathrm{char}\, k = 0$ if $d > 3$.

We fix a very ample divisor D on X and consider the linear systems $|mD|$ for $m \in \mathbf{N}$. Every such $|mD|$ can be considered as the set of closed points of a projective space, and we will indeed identify $|mD|$ with the corresponding projective space over k.

Theorem 1. *For any $m \in \mathbf{N}$, the geometrically integral divisors in the linear system $|mD|$ form an open subvariety $\mathcal{P}_{mD}$. For any conjugacy class $\mathcal{C}$ of a subgroup H of G, there is a locally closed subvariety $\mathcal{D}^{\mathcal{C}}_{mD}$ consisting of those divisors in $\mathcal{P}_{mD}$ of decomposition type $\mathcal{C}$, and*

$$\limsup_{m\to\infty} \frac{\dim \mathcal{D}^{\mathcal{C}}_{mD}}{\dim \mathcal{P}_{mD}} = \frac{1}{[G:H]^{d-1}}.$$

Moreover, this limit inferior becomes a limit if D (or any linearly equivalent prime divisor) splits completely in Y.

In particular, for every subgroup H of G there are infinitely many Weil prime divisors on Y with decomposition group H. Furthermore, for fixed X, one can deduce that a finite branched Galois cover $f : Y \to X$ is completely described by the set of Weil prime divisors that split completely.

One side note: The more precise description of $\mathcal{P}_{mD}$ is that for *every* field extension $k'|k$, $\mathcal{P}_{mD}(k')$ consists exactly of those effective divisors on $X' := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$ which are linearly equivalent to the base change D' of D to X'. Similarly, one describes $\mathcal{D}^{\mathcal{C}}_{mD}$. This way, the scheme structures and hence dimensions of $\mathcal{P}_{mD}$ and $\mathcal{D}^{\mathcal{C}}_{mD}$ are indeed uniquely defined.

2. Special Case: $k = \mathbf{F}_q$

In the case where k is a finite field, the sets $\mathcal{P}_{mD}(k)$, $\mathcal{D}_{mD}(k)$ are finite, and we can actually *count* divisors:

Theorem 2. *Under the assumptions from above, let k be a finite field. Then*

$$\limsup_{m\to\infty} \frac{\log \#\mathcal{D}^{\mathcal{C}}_{mD}(k)}{\log \#\mathcal{P}_{mD}(k)} = \frac{1}{[G:H]^{d-1}}.$$

Both theorems are proven in a similar manner using considerations on the behavior of volumes of divisors under pullback and push-forward. The only major difference of the two proofs is that the first one uses the classical Bertini theorem whereas the second one use Poonen's Bertini theorem over finite fields ([2]).

3. Connection with a Result of F.K. Schmidt

The above-mentioned statements can also be reinterpreted as giving effective versions of (a special case of) a result of F.K. Schmidt ([3]):

Theorem 3 (F.K. Schmidt)**.** *Suppose Ω is a Hilbertian field, and $K|\Omega$ is a separably generated function field in one variable. Let $L|K$ be a finite Galois extension. Then for any subgroup H of* $\mathrm{Gal}(L|K)$*, there are infinitely many valuations on L which are constant on Ω and have decomposition group H.*

An important case of Hilbertian fields are function fields. For these fields, our theorem can be used to describe more explicitly "how often" a particular subgroup H actually occurs as a decomposition group, at least under some mild additional assumptions:

Assume Ω itself is a function field in one variable over a perfect field k, i.e. L and K are both function fields in two variables over k; and assume k is relatively algebraically closed in L. Then we can choose a normal projective model X/k for $K|k$ and take its normalization Y in L to get a finite branched Galois cover $f : Y \to X$ of two-dimensional, normal, geometrically integral projective k-varieties. F.K. Schmidt's theorem follows from ours by identifying Weil prime divisors on Y with the corresponding valuations.

References

[1] A. Holschbach, *A Chebotarev-like Density Theorem in Algebraic Geometry*, Ph.D. Thesis, University of Pennsylvania (2008).

[2] B. Poonen, *Bertini Theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127.

[3] F.K. Schmidt, *Über die Kennzeichnung algebraischer Funktionenkörper durch ihren Regularitätsbereich*, J. Reine Angew. Math. **171** (1934), 162–169.

[4] J.-P. Serre, *Zeta and L functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), 82–92. Harper & Row, New York, 1965.