A Chebotarev-like Density Theorem in Algebraic Geometry

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A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2008

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Acknowledgments

First and foremost, let me express my deepest gratitude to my thesis advisor Florian Pop. His constant guidance, support and suggestions not only led me through my path towards the Ph.D., but broadened my mathematical horizon in general. Many thanks also to Gerda and the whole family for all the hospitality and help I enjoyed.

I also would like to thank everybody at the Department of Mathematics for the excellent education I was able to experience. My special thanks go to David Harbater, Ted Chinburg and Steve Shatz, whose suggestions helped a lot while working on my thesis. Of course, I should not forget to express my thanks to Janet, Monica, Paula and Robin in the Math Office, who are doing a magnificent job.

My time in Philadelphia would not have been as much fun without all my friends and colleagues, like Toby, Cisca, Shuvra, Pranav, Martin, David, Jen, Chris, Asher, Andrew, John and Pilar, to name only a few. I will miss all of you and hope to see you again sooner rather than later. Also, I would like to thank Shuvra, Pranav and Toby for all the helpful mathematical discussions we have had.

I am grateful to my mother and my whole family for keeping me grounded and encouraging me all the time.

Last but not least, I want to express my sincere gratitude and love to my girlfriend Verena Schwarzer, who gave me all the love, care and support I needed and so much more.

ABSTRACT

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For an étale Galois cover of geometrically normal, geometrically integral projective varieties of dimension $d \geq 2$ over an arbitrary field k, we prove a Chebotarev-like density theorem which describes the decomposition behavior of prime divisors. In characteristic zero, the étaleness condition can be dropped. As an application, we give a numerical criterion for such a cover to be Galois.

Contents

Chapter 1

Introduction

In 1922, Nikolai Chebotarev proved his famous density theorem, which had been conjectured in 1880 by Frobenius. It describes the decomposition behavior of prime ideals in Galois extensions. Let us explain the context:

Let $L|K$ be a finite Galois extension of number fields with Galois group G, and let $\mathcal{O}_L|\mathcal{O}_K$ be the corresponding rings of integers. Let $\mathfrak{p} \subset \mathcal{O}_K$ a prime ideal unramified in $\mathcal{O}_L$, and let $\mathfrak{q} \subset \mathcal{O}_L$ be a prime ideal lying above $\mathfrak{p}$. Then the decomposition group $G_{\mathfrak{q}} = \{\sigma \in G | \sigma\mathfrak{q} = \mathfrak{q}\}$ is canonically isomorphic to the Galois group $\mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}|\mathcal{O}_K/\mathfrak{p})$. The latter group contains a distinguished element: the Frobenius automorphism, which acts on $\mathcal{O}_L/\mathfrak{q}$ by $x \mapsto x^{N\mathfrak{p}}$, where $N\mathfrak{p} := \#\mathcal{O}_K/\mathfrak{p}$. Via the aforementioned canonical isomorphism, we can regard the Frobenius of $\mathfrak{q}$ as an element of G; its conjugacy class depends only on $\mathfrak{p}$ and is denoted by $\left(\frac{L|K}{\mathfrak{p}}\right)$.

Recall that for a set $\mathcal{P}$ of primes of K, we define its Dirichlet density to be

$$d(\mathcal{P}) = \lim_{s \searrow 1} \frac{\sum_{\mathfrak{p} \in \mathcal{P}} N\mathfrak{p}^{-s}}{\sum_{\mathfrak{p}} N\mathfrak{p}^{-s}}, \qquad \text{provided the limit exists.}$$

Theorem (Chebotarev's density theorem)**.** *Let $\mathcal{C}$ be the conjugacy class of an element of G. Set $D^{\mathcal{C}}_{L|K} = \{\mathfrak{p} \subset \mathcal{O}_K | \mathfrak{p}$ unramified in $\mathcal{O}_L, \left(\frac{L|K}{\mathfrak{p}}\right) = \mathcal{C}\}$. Then $D^{\mathcal{C}}_{L|K}$ has a Dirichlet density, precisely:*

$$d(D^{\mathcal{C}}_{L|K}) = \frac{\#\mathcal{C}}{\operatorname{ord} G}.$$

This theorem (and its original proof) marked one of the cornerstones of number theory; among other things, it enabled Artin to prove his reciprocity law. Later, the density theorem was generalized in various ways; the most general treatment is that of Serre in [Se65]: He replaces the Galois extension of number fields by a generically finite Galois cover of schemes of finite type over $\mathbb{Z}$, and the prime ideals are replaced by the closed points of these schemes. The rest of the statement (and the basic idea of the proof) still stay more or less the same.

Serre's version of Chebotarev's density theorem therefore describes the decomposition behavior of closed points, for example for varieties over finite fields. But higher dimensional varieties contain more than just these closed points (and the generic point). Thus, a very natural question to ask is whether one can give comparable density results for the behavior of higher dimensional points on these varieties, like curves or divisors.

In this thesis we will consider the decomposition behavior of *divisors* in Galois covers of varieties. Let us fix some notation first: Let $g : Z \to X$ be a (branched) Galois cover, i.e. a finite dominant morphism together with a finite group G acting on Z such that X is (isomorphic to) the quotient variety Z/G. For any Weil prime divisor F on Z, we set the decomposition group G_F of F to be $\{\sigma \in G | \sigma F = F\}$. For a Weil prime divisor D on X, we define the decomposition class $\mathcal{C}_D$ of D to be the conjugacy class of G_F, where F is a Weil prime divisor on Z with $g(F) = D$.[1] Note that the notion of $\mathcal{C}_D$ depends only on D, not on the choice of F over D.

As it turns out, it is possible to come up with a notion of density which gives Chebotarev-like results for projective varieties over any field, not just finite fields. Let us start with one version which gives the least restrictions on the cover $Z \to X$, but will be proven in characteristic zero only:

Theorem A (3.6.1)**.** *Let $g : Z \to X$ be a finite branched Galois cover with Galois group G of normal, geometrically integral projective varieties over a field k of characteristic zero, and assume* $\dim X = d \geq 2$. *Let $\mathcal{C}$ be a conjugacy class of subgroups of G, and let D be an ample divisor on X. Then for every $m \in \mathbb{N}$, there are varieties $\mathbf{P}_{mD}$ and $\mathbf{D}^{\mathcal{C}}_{mD}$ representing the geometrically integral divisors on X which are linearly equivalent to mD, respectively those which additionally are unramified in the cover $Z \to X$ and have decomposition class $\mathcal{C}$.*

Assume furthermore that D is c-split *in Z, i.e., there exists a Cartier divisor F on Z with $g(F) = D$ and $k(F) = k(D)$. Then*

$$\lim_{m\to\infty} \frac{\dim \mathbf{D}^{\mathcal{C}}_{mD}}{\dim \mathbf{P}_{mD}} = \frac{1}{[G : \mathcal{C}]^{d-1}},$$

where $[G : \mathcal{C}]$ is defined to be $[G : H]$ for any representative H of $\mathcal{C}$. Without the assumption of D being c-split, the statement still holds if we replace D by $\mathrm{ord}(G) \cdot D$ *or regard the limit superior instead of the limit.*

In this theorem, the representability statement about $\mathbf{P}_{mD}$ and $\mathbf{D}^{\mathcal{C}}_{mD}$ means that for any arbitrary field extension $K|k$, the geometrically integral divisors on $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ which are linearly equivalent to mD_K correspond to the K-rational points of $\mathbf{P}_{mD}$, and similarly for $\mathbf{D}^{\mathcal{C}}_{mD}$. In order to get an easier understanding of what this means, we can consider a special case:

[1] Since the decomposition groups are noncyclic in general, we cannot expect to single out one element to play the role of a Frobenius, hence the change to conjugacy classes of subgroups.

Corollary. *Under the assumptions of Theorem A, assume furthermore that k is algebraically closed. Then for any $m \in \mathbb{N}$, the set of all Weil prime divisors in the complete linear system $|mD|$ form (the closed points of) a variety $\mathbf{P}_{mD}$.*[2] *Also, the set of all Weil prime divisors in the linear system $|mD|$ with decomposition class $\mathcal{C}$ form (the closed points of) a subvariety $\mathbf{D}^{\mathcal{C}}_{mD}$, and the quotient of the dimensions of these two varieties converges for $m \to \infty$ as described above.*

Actually, we can extend this theorem to varieties over fields of arbitrary characteristic if we put some more restrictions on the cover:

Theorem B (4.1.4)**.** *Let $g : Z \to X$ be a finite* étale *Galois cover with Galois group G of geometrically normal, geometrically integral projective varieties over an arbitrary field k, and assume $\dim X = d \geq 2$. Let $\mathcal{C}$ be a conjugacy class of subgroups of G, and let D be an ample divisor on X. Then for every $m \in \mathbb{N}$, there are varieties $\mathbf{P}^{\mathbf{gn}}_{mD}$ and $\mathbf{D}^{\mathbf{gn},\mathcal{C}}_{mD}$ representing the geometrically normal, geometrically integral divisors on X which are linearly equivalent to mD, respectively those which additionally have decomposition class $\mathcal{C}$. Then*

$$\limsup_{m\to\infty} \frac{\dim \mathbf{D}^{\mathbf{gn},\mathcal{C}}_{mD}}{\dim \mathbf{P}^{\mathbf{gn}}_{mD}} = \frac{1}{[G : \mathcal{C}]^{d-1}}.$$

If D (or any divisor in $|D|$) is c-split in Z, then $\lim_{m\to\infty} \frac{\dim \mathbf{D}^{\mathbf{gn},\mathcal{C}}_{mD}}{\dim \mathbf{P}^{\mathbf{gn}}_{mD}}$ exists, and is equal to $\frac{1}{[G:\mathcal{C}]^{d-1}}$.

For finite fields, the divisors in a linear system can actually be counted, and we get another version of Chebotarev-like density theorem:

Theorem C (4.2.5)**.** *Let $g : Z \to X$ be a finite étale Galois cover with Galois group G of geometrically normal, geometrically integral projective varieties over a finite field k, and assume $\dim X = d \geq 2$. Let $\mathcal{C}$ be a conjugacy class of subgroups of G, and let D be an ample divisor on X. Then for every $m \in \mathbb{N}$, let $p_{\#}(mD)$ and $dc^{\mathcal{C}}_{\#}(mD)$ denote the number of geometrically normal, geometrically integral divisors in $|mD|$, respectively the number of those which additionally have decomposition class $\mathcal{C}$. Then*

$$\limsup_{m\to\infty} \frac{\log dc^{\mathcal{C}}_{\#}(mD)}{\log p_{\#}(mD)} = \frac{1}{[G : \mathcal{C}]^{d-1}}.$$

Again, if D (or any divisor in $|D|$) is c-split in Z, then $\lim_{m\to\infty} \frac{\log dc^{\mathcal{C}}_{\#}(mD)}{\log p_{\#}(mD)}$ exists, and is equal to $\frac{1}{[G:\mathcal{C}]^{d-1}}$.

Of course, the last result is not entirely surprising once one knows theorem B, since at least for a projective space $\mathbb{P}$ over a finite field $k = \mathbb{F}_q$, the number of

[2] Actually, it will be an open subvariety of $\mathbb{P}H^0(X, mD)$.

rational points is approximately $q^{\dim \mathbb{P}}$ (up to some bounded factor), so it does not seem unreasonable that $\log_q p_\#(mD) = \log_q \#\mathbf{P}^{\mathbf{gn}}_{mD}(k)$ comes close to $\dim \mathbf{P}^{\mathbf{gn}}_{mD}$. Still, the proof will be more than just a simple variation of the proofs of theorems A and B.

As an application, let us mention the following theorem:

Theorem (5.1.7)**.** *Let $f : Y \to X$ be a finite (branched) cover of geometrically normal, geometrically integral projective varieties of dimension $d \geq 2$ over an arbitrary field k, and assume f to be étale if* $\operatorname{char} k > 0$. *We say that a divisor D on X* (partially) splits *in Y if there exists a Weil prime divisor E on Y with $f(E) = D$ and $[k(E) : k(D)] = 1$; we call D* completely split *in Y if there are* $\deg(f)$ *different divisors E over D with this property. Then the following are equivalent:*

- *Every Weil prime divisor D on X that is unramified and splits in Y is completely split in Y.*
- *$f : Y \to X$ is a Galois cover.*

We will now present a short outline of the proof of theorem A; the difference in the proof of theorem B will be mentioned at the point where it occurs.

Concerning the existence and asymptotic behavior of the dimension of $\mathbf{P}_{mD}$, the argument is rather simple: A Bertini-type argument proves $\mathbf{P}_{mD}$ to be an open dense subvariety of $\mathbb{P}H^0(X, mD)$ (3.4.4), and an asymptotic Riemann-Roch theorem (3.3.1) then describes the asymptotic behavior of its dimension.

So we can concentrate on $\mathbf{D}^{\mathcal{C}}_{mD}$. Fix a representative H of $\mathcal{C}$, set $Y = Z/H$ and let $h : Z \to Y$ and $f : Y \to X$ be the corresponding quotient morphisms. Then one can show that $\mathbf{D}^{\mathcal{C}}_{mD}$ is an open dense subscheme of the scheme $\mathbf{S}_{mD,Y}$ representing all geometrically integral divisors linearly equivalent to mD which split in Y, i.e. which are push-forwards f_*E of effective Weil divisors E on Y.[3]

As it turns out, it is much easier to find a scheme $\mathbf{Sc}_{mD,Y}$ representing the geometrically integral divisors linearly equivalent to mD which are *c-split* in Y, i.e. which are push-forwards f_*E of effective *Cartier* divisors E on Y.

The strategy how to construct $\mathbf{S}_{mD,Y}$ from $\mathbf{Sc}_{mD,Y}$ is where the argument differs in the proofs of theorems A and B. For varieties over fields of characteristic 0, we use a resolution of singularities to reduce to the case where Y is nonsingular (3.5.1); then the notions of Cartier and Weil divisor coincide, and we can identify the two schemes. In the context of theorem B, we use that fact that f is étale to show that the notions of split and c-split coincide (4.1.2).

In order to get $\mathbf{Sc}_{mD,Y}$, we use the classical result of Grothendieck that the (relative) effective Cartier divisors on X are representable by a scheme $\mathbf{Div}_{X/k}$, and that the push-forward map $f_* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ on Cartier divisors gives

[3]This notion of splitting coincides with the one mentioned before, as we will see later.

rise to a (locally) finite morphism $\mathbf{f}_* : \mathbf{Div}_{Y/k} \to \mathbf{Div}_{X/k}$. Then $\mathbf{Sc}_{mD,Y}$ is just the intersection of $\mathbf{P}_{mD}$ with $\mathbf{f}_*(\mathbf{Div}_{Y/k})$, and its dimension equals the dimension of the preimage of $\mathbf{P}_{mD}$ under $\mathbf{f}_*$.

From there it is only a short step to see that it suffices to give asymptotic bounds for the dimension of the scheme $\mathbf{Div}_{Y/k}^{[E]_{num}}$ representing the relative effective divisors which are numerically equivalent to a given Cartier divisor E on Y, with f_*E linearly equivalent to mD (3.2.4), and this simplifies even further to compare $h^0(Y, E)$ of such E asymptotically with $h^0(X, mD)$, which is taken care of in section 3.3.

Structure of the Manuscript.

- In chapter 2 we recall some facts on divisors, their representability, and intersection theory.
- The proof of Theorem A is contained in chapter 3: First we construct the push-forward morphism $\mathbf{f}_*$, then we use it to define and represent c-split divisors. After an excursion to the volume of divisors and their behavior in finite covers, we use these facts to give upper and lower bounds on the asymptotics of $\dim \mathbf{S}_{mD,Y}$, which are then used to prove theorem A.
- Chapter 4 covers the proofs of theorems B and C; the main novelty in the proof of theorem C is that we have to use a different Bertini's theorem by Poonen which holds over finite fields.
- Finally, in chapter 5 we give applications of this theory, which are mainly centered around the theorem mentioned above.

Notation.

If X and T are varieties over a given field k, we denote by X_T the fiber product $X \times_{\operatorname{Spec} k} T$, which is regarded as a base change of X to a scheme over T. Similarly, if K is a field extension of k, then X_K denotes the K-variety $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$.

If D is a Cartier divisor on a projective k-variety X, we usually write $H^0(X, D)$ for $H^0(X, \mathcal{O}_X(D))$ and $h^0(X, D)$ for $\dim_k H^0(X, D)$. The same holds for higher cohomology.

$\bar{k}$ always denotes an algebraic closure of the field k.

Chapter 2

Preliminaries

The following will contain some base information about divisors, invertible sheaves and equivalence classes of such that will be need later. For a more comprehensive account of this matter, we refer to Kleiman's excellent article in [FAG05]. Most of the results are well-known for nonsingular varieties over algebraically closed fields, but since we were aiming for more general results, we included them for the convenience of the reader.

2.1 Divisors

Let X be a normal geometrically integral projective variety over a field k. We can define a functor $\mathrm{Div}_{X/k}$ by setting

$$\mathrm{Div}_{X/k}(T) := \left\{ \begin{array}{l} \text{relative effective divisors } D \text{ on } X_T/T, \\ \text{i.e. effective (Cartier) divisors } D \text{ on } X_T \text{ that are } T\text{-flat} \end{array} \right\}.$$

Alternatively, one can describe the relative effective divisors on X_T/T as the subschemes $D \subset X_T$ for which the following holds: For any $x \in X_T$, t its image in T, D is cut out at x by one element that is regular on the fiber $X_{\kappa(t)}$ ([FAG05, 9.3.4]).

This functor is representable by an open subscheme $\mathbf{Div}_{X/k}$ of the Hilbert scheme $\mathbf{Hilb}_{X/k}$ ([FAG05, Thm 9.3.7]).

Lemma 2.1.1. *Every connected component of* $\mathbf{Div}_{X/k}$ *is proper over* k.

Proof. Using the valuative criterion of properness, all we have to show is that for a discrete valuation ring R over k with field of fractions K, the induced map $p^* : \mathbf{Div}_{X/k}(R) \to \mathbf{Div}_{X/k}(K)$ induced by $p : \operatorname{Spec} K \to \operatorname{Spec} R$ is an isomorphism. Since we know that every connected component of the Hilbert scheme is proper ([FAG05, 5.1.5.(7)]), it suffices to prove that for every coherent sheaf $\mathcal{F}$ on $X_R = X \times_{\operatorname{Spec} k} \operatorname{Spec} R$ which is a quotient of $\mathcal{O}_{X_R}$, $\mathcal{F}$ is a quotient of $\mathcal{O}_{X_R}$ by an invertible sheaf iff $p^*\mathcal{F}$ is a quotient of $\mathcal{O}_{X_K}$ by an invertible sheaf. Defining $\mathcal{I}$ to be the kernel of the surjection $\mathcal{O}_{X_R} \to \mathcal{F}$ and using the flatness of $p : \operatorname{Spec} K \to \operatorname{Spec} R$, this translates into showing the following: An ideal sheaf $\mathcal{I}$ on X_R such that $V(\mathcal{I})$ is flat over $\operatorname{Spec} R$ is invertible iff $p^*\mathcal{I}$ is.

Since this can be checked locally, we can restrict ourselves to the case that $X = \operatorname{Spec} A$ is affine, where A is a finitely generated k-algebra, and $\mathcal{I} = \tilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subseteq A_R := A \otimes_k R$. We have to prove that $\mathfrak{a}$ is a principal ideal whenever $\mathfrak{a}_K = \mathfrak{a} \otimes_R K$ is. This can be seen the following way: The discrete valuation $v : K \to \mathbb{Z}$ corresponding to R extends canonically to a discrete valuation $v_A : A_K := A \otimes_k K \to \mathbb{Z}$ with the property that $v_A^{-1}(\mathbb{N}) = A_R \subset A_K$; this implies that if we have two elements $f, g \in A_R$ with $v_A(f) \leq v_A(g)$ and f divides g in A_K, then f divides g in A_R. If $\mathfrak{a} \subset A_R$ is an ideal such that $A_R/\mathfrak{a}$ is flat over R and such that $\mathfrak{a}_K$ is principal, take a generator f of $\mathfrak{a}_K$. Without loss of generality, we can assume $v(f) = 0$. Then since $A_R/\mathfrak{a}$ is torsionfree and $\pi^n f \in \mathfrak{a}$ for $n \in \mathbb{N}$, π a uniformizer of R, we have $f \in \mathfrak{a}$. Hence, by what we said before, f generates $\mathfrak{a}$. This completes the proof. $\square$

2.2 Picard scheme and linear systems

Let $\operatorname{Pic}(X)$ be the Picard group of invertible sheaves. There is natural morphism $\operatorname{Div}(X) \to \operatorname{Pic}(X)$ sending a divisor D to the sheaf $\mathcal{O}_X(D)$; its kernel consists exactly of the principal divisors.

Similarly to $\operatorname{Div}_{X/k}$, one can define a *relative Picard functor* $\operatorname{Pic}_{X/k}$ by

$$\operatorname{Pic}_{X/k}(T) := \operatorname{Pic}(X_T)/\operatorname{Pic}(T).$$

Furthermore, let $\operatorname{Pic}_{(X/k)(\mathrm{fppf})}$ be the associated sheaf in the fppf topology. The first functor injects naturally into the latter ([FAG05, 9.2.2, 9.2.5, 9.3.11]). $\operatorname{Pic}_{(X/k)(\mathrm{fppf})}$ is representable by a separated group scheme, the Picard scheme $\mathbf{Pic}_{X/S}$ ([FAG05, 9.3.18.3]). The previously prescribed map of functors $A_{X/k} : \operatorname{Div}_{X/k} \to \operatorname{Pic}_{X/k}$ induces the so-called *Abel map* of schemes $\mathbf{A}_{X/k} : \mathbf{Div}_{X/k} \to \mathbf{Pic}_{X/k}$. The Abel map is proper, since X is geometrically integral ([FAG05, 9.4.12]).

For an invertible sheaf $\mathcal{L}$ on X, define a subfunctor $\operatorname{LinSys}_{\mathcal{L}/X/k}$ of $\operatorname{Div}_{X/S}$ by

$$\operatorname{LinSys}_{\mathcal{L}/X/k}(T) := \left\{ \begin{array}{l} \text{relative effective divisors } D' \text{ on } X_T/T \text{ such that} \\ \mathcal{O}_{X_T}(D') \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N} \text{ for some invertible sheaf } \mathcal{N} \text{ on } T \end{array} \right\}.$$

Then $\operatorname{LinSys}_{\mathcal{L}/X/k}$ is representable by a projective *space* $\mathbf{L}_{\mathcal{L}/X/k}$ over k ([FAG05, 9.3.13]). $\mathbf{L}_{\mathcal{L}/X/k}$ can also be regarded as the fiber of the Abel map over $\mathcal{L} \in \mathbf{Pic}_{X/k}$. Its dimension is $h^0(X, \mathcal{L}) - 1$. For a Cartier divisor D on X, we denote $\operatorname{LinSys}_{\mathcal{O}_X(D)/X/k}$ by $\operatorname{LinSys}_{D/X/k}$ and $\mathbf{L}_{\mathcal{O}_X(D)/X/k}$ by $\mathbf{L}_{D/X/k}$ or simply $\mathbf{L}_D$, if no confusion can arise.

2.3 Algebraic and numerical equivalence

Let $\mathbf{Pic}^0_{X/k}$ denote the connected component of the identity inside the group scheme $\mathbf{Pic}_{X/k}$. It is a geometrically irreducible open and closed group subscheme of finite type ([FAG05, 9.5.3]). For an invertible sheaf $\mathcal{L}$ on X, the corresponding point $\lambda \in \mathbf{Pic}_{X/k}$ lies in $\mathbf{Pic}^0_{X/k}$ if and only if $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$ ([FAG05, 9.5.10]), i.e. if there exist connected k-schemes T_i of finite type, $i = 1, \ldots, n$ for some n, geometric points s_i, t_i of T_i with the same residue field, and invertible sheaves $\mathcal{M}_i$ on X_{T_i} such that

$$\mathcal{L}_{s_1} \simeq \mathcal{M}_{1,s_1}, \mathcal{M}_{1,t_1} \simeq \mathcal{M}_{2,s_2}, \ldots, \mathcal{M}_{n-1,t_{n-1}} \simeq \mathcal{M}_{n,s_n}, \mathcal{M}_{n,t_n} \simeq \mathcal{O}_{X_{t_n}}.$$

For a (Cartier) divisor D and a curve $C \subset X$, define $D \cdot C = \deg(\mathcal{O}_C(D))$ and extend by linearity to an intersection pairing between divisors and 1-cycles. Two divisors D_1, D_2 are said to be numerically equivalent, $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every 1-cycle C. A divisor is D is numerically equivalent to 0 if and only if $\mathcal{O}_X(mD)$ is algebraically equivalent to $\mathcal{O}_X$ for some nonzero m ([FAG05, 9.6.3]).

By what we have said before, it is clear that D is numerically trivial if and only the corresponding point $\lambda \in \mathbf{Pic}_{X/k}$ lies in $\mathbf{Pic}^\tau_{X/k} = \bigcup_{n>0} \varphi_n^{-1} \mathbf{Pic}^0_{X/k}$, where φ_n is the n-th power map. $\mathbf{Pic}^\tau_{X/k}$ is an open and closed subgroup scheme of finite type ([FAG05, 9.6.12]).

Denote the numerical equivalence class of a divisor D on X by $[D]_{num}$, and let $\mathbf{Div}^{[D]_{num}}_{X/k}$ denote the preimage of $\mathbf{Pic}^{[D]_{num}}_{X/k} := \mathcal{O}_X(D) + \mathbf{Pic}^\tau_{X/k}$ under the Abel map. For any Cartier divisor D on X, both $\mathbf{Div}^{[D]_{num}}_{X/k}$ and $\mathbf{Pic}^{[D]_{num}}_{X/k}$ are finitely generated over k: the latter one is just a translate of $\mathbf{Pic}^\tau_{X/k}$, and the statement for the first one follows since the Abel map is proper.

The Neron-Severi theorem states that the group $\operatorname{Pic}_{X_{\bar{k}}/\bar{k}}(\bar{k}) / \operatorname{Pic}^0_{X_{\bar{k}}/\bar{k}}(\bar{k})$ is finitely generated ([SGA6, XIII.5.1]). By the above, this implies that the group of Cartier divisors on X modulo algebraic equivalence is finitely generated; moreover, the Neron-Severi group $N^1(X)$ of Cartier divisors on X modulo numerical equivalence is finitely generated and free abelian.

2.4 Numerical criteria for divisors

The intersection product for divisors and 1-cycles mentioned above can be extended to a more general intersection product: If $D_1, \ldots, D_r$ are Cartier divisors on X with $r \geq \dim X$, define the intersection number

$$D_1 \cdots D_r = \int_X D_1 \cdots D_r$$

to be the coefficient of the monomial $m_1 \cdots m_r$ in the multivariate Hilbert polynomial $\chi(X, m_1 D_1 + \ldots + m_r D_r)$. For a subscheme V of X of dimension at most s,

set
$$D_1 \cdots D_s \cdot V = \int_V D_1|_V \cdots D_s|_V.$$
This notion of an intersection product is multilinear, symmetric, takes integral values and comprises the above definition of intersections between divisors and curves ([De01, 1.8]). Furthermore, all of these products depend only on the numerical classes of the divisors ([La04a, 1.1.18]).

The Nakai-Moishezon criterion gives a numerical description of ample divisors: A Cartier divisor D is ample if and only if for every integral subscheme V of X, one has $D^{\dim(V)} \cdot V > 0$. In the wake of this theorem, one defines a Cartier divisor to be *nef* iff for every integral subscheme V of X, one has $D^{\dim(V)} \cdot V \geq 0$. In fact, it is sufficient to check this inequality for all integral curves on X ([De01, 1.26]).

It is useful to extend the Neron-Severi group $N^1(X)$ to a real vector space $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and view $N^1(X)$ as a complete lattice inside it. If one extends the definition of the intersection products by linearity, then the numerical description of ampleness and nefness allows to extend these notions to $N^1(X)_{\mathbb{R}}$; let $\mathrm{Amp}(X) \subset N^1(X)$ denote the cone of all ample divisor classes, $\mathrm{Nef}(X)$ the nef cone. If we define $N_1(X)$ to be the group of all 1-cycles modulo numerical equivalence, and let $\overline{NE}(X) \subset N_1(X)_{\mathbb{R}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the closure of the cone of curves (generated by the effective 1-cycles), then $\mathrm{Nef}(X)$ and $\overline{NE}(X)$ are dual to each other. By Kleiman's criterion ([De01, 1.27]), $\delta \in N^1(X)_{\mathbb{R}}$ is ample if and only if $\delta \cdot z > 0$ for every $z \in \overline{NE}(X)$, so the ample cone is open, and its closure is the nef cone.

A Cartier divisor D on X is called *big* if $\liminf_{m\to\infty} \frac{h^0(X,mD)}{m^{\dim X}} > 0$ ([De01, 1.30]). A divisor is big if and only if some positive multiple of it is numerically equivalent to the sum of an ample and an effective divisor ([La04a, 2.2.7]). In particular, the notion is stable under numerical equivalence, so it makes sense to speak of big classes in $N^1(X)$. More general, one defines an element $\delta \in N^1(X)_{\mathbb{R}}$ to be big if it can be written in the form $\delta = \sum_{i=1}^r a_i \delta_i$ with $a_i > 0$ and $\delta_i \in N^1(X)$ big ([La04a, 2.2.21]). The set of all big $\delta \in N^1(X)_{\mathbb{R}}$ is the *big cone* $\mathrm{Big}(X)$; it is open and convex and contains the ample cone. Its closure is the *pseudoeffective cone* $\overline{\mathrm{Eff}}(X)$, which is defined to be the closure of the convex cone generated by the classes of effective divisors ([La04a, 2.2.26]).

Lemma 2.4.1. *The pseudoeffective cone has a compact basis, i.e. for $\delta \in \overline{\mathrm{Eff}}(X)$, the set $\{\delta' \in \overline{\mathrm{Eff}}(X) | \delta - \delta' \in \overline{\mathrm{Eff}}(X)\}$ is compact.*

Proof. If $d := \dim(X)$ is at most 1, the statement is trivial, since $N^1(X)$ is either trivial or $\mathbb{Z}$. Assume therefore that $d \geq 2$. Let $\eta_1, \cdots, \eta_\rho$ a basis of $N^1(X)_{\mathbb{R}}$ consisting of ample classes. Then for every $i = 1, \ldots, \rho$, and every $\delta \in N^1(X)_{\mathbb{R}}$, set
$$\deg_i(\delta) := \delta \cdot \eta_i \cdot \eta_1^{d-2}$$
The $\deg_i$, $i = 1, \ldots, \rho$, form a basis of the dual space of $N^1(X)_{\mathbb{R}}$. Indeed, if we assume $\deg_i(\delta) = 0 \ \forall \ i$, by linearity it follows that $\delta^2 \cdot \eta_1^{d-2} = \delta \cdot \eta_1^{d-1} = 0$,

which implies $\delta = 0$ by [FAG05, 9.6.3 (h)$\Rightarrow$(b)] (the main argument is the Hodge index theorem). Now for $\delta \in \overline{\mathrm{Eff}}(X)$, we have $\deg_i(\delta) \geq 0 \ \forall \ i$ (if $\delta, \eta_1, \ldots, \eta_\rho$ are classes corresponding to integral Cartier divisors, this follows from Nakai-Moishezon criterion; it extends by linearity and continuity). This implies that for fixed δ, the closed set $\{\delta' \in \overline{\mathrm{Eff}}(X) | \delta - \delta' \in \overline{\mathrm{Eff}}(X)\}$ is a subset of the compact set $\{\delta' \in \overline{\mathrm{Eff}}(X) | 0 \leq \deg_i(\delta') \leq \deg_i(\delta) \text{ for } i = 1, \ldots, \rho\}$, hence is compact itself. $\square$

Corollary 2.4.2. *Define a partial order on $N^1(X)$ by setting $\delta \geq \delta'$ if the class $\delta - \delta'$ can be represented by an effective divisor. Then for every given effective divisor D, there are only finitely many classes of effective divisors which are smaller than $[D]_{num}$.*

Chapter 3

Behavior of divisors in finite covers

3.1 Finite morphisms and induced maps on divisors

Let X, Y be normal geometrically integral projective varieties over a field k.

3.1.1 Pull-back

For a generically finite morphism $f : Y \to X$, we define pull-back morphisms $\mathbf{f}^* : \mathbf{Div}_{X/k} \to \mathbf{Div}_{Y/k}$ and $\mathbf{f}^* : \mathbf{Pic}_{X/k} \to \mathbf{Pic}_{Y/k}$. These morphisms fit into a commutative diagram

$$\begin{tikzcd}
\mathbf{Div}_{X/k} \arrow[r, "\mathbf{f}^*"] \arrow[d, "\mathbf{A}_{X/k}"'] & \mathbf{Div}_{Y/k} \arrow[d, "\mathbf{A}_{Y/k}"] \\
\mathbf{Pic}_{X/k} \arrow[r, "\mathbf{f}^*"] & \mathbf{Pic}_{Y/k}
\end{tikzcd}$$

The existence of these morphisms and the commutativity of the diagram follows from the construction of natural transformations $f^* : \mathrm{Div}_{X/k} \to \mathrm{Div}_{Y/k}$ and $f^* : \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{Y/k}$ and their compatibility.

In fact, both these transformations are well-known: For a invertible sheaf $\mathcal{L}$ on X_T, the sheaf-theoretic pull-back $f^*\mathcal{L}$ is an invertible sheaf on Y_T; the induced map is $f^* : \mathrm{Pic}(X_T) \to \mathrm{Pic}(Y_T)$ is a group homomorphism, behaves functorially and hence gives rise to a natural transformation $f^* : \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{Y/k}$.

Similarly, for any relative effective Cartier divisor $D \in \mathrm{Pic}_{X/k}(T)$, D is given by a collection of (U_i, r_i), where $\mathfrak{U} = (U_i)_i$ is an open covering of X_T, and $r_i \in \mathcal{O}_{X_T}(U_i)$ such that

- $r_j|_{U_i \cap U_j} \in r_i|_{U_i \cap U_j} \mathcal{O}_{X_T}(U_i \cap U_j)^\times \;\forall\, i, j$,
- for all i, r_i is regular on $X_{\kappa(t)}$ for any $t \in \mathrm{pr}_2(U_i) \subset T$.

Now we claim that the collection of $(f^{-1}(U_i), f^{\#}(r_i))$ corresponds to a relative effective divisor $f^*D \in \mathrm{Pic}_{Y/k}(T)$.

In fact, the only fact that is not obvious is that for all i, $f^{\#}(r_i)$ is regular on $Y_{\kappa(t)}$ for every $t \in \mathrm{pr}_2(U_i)$. But since we assumed X and Y to be geometrically integral, all $X_{\kappa(t)}$, $Y_{\kappa(t)}$ are integral, so all we have to show the image $\overline{f^{\#}(r_i)}$ of $f^{\#}(r_i)$ in $K(Y_{\kappa(t)})$ is nonzero for any $t \in \mathrm{pr}_2(U_i)$. But this is trivial, since for any such t, $\overline{f^{\#}(r_i)}$ is the image of $\bar{r}_i$ under the imbedding $K(X_{\kappa(t)}) \hookrightarrow K(Y_{\kappa(t)})$, and $\bar{r}_i$ is nonzero, since regular.

Obviously, this gives a natural transformation $f^* : \mathrm{Div}_{X/k} \to \mathrm{Div}_{Y/k}$ which preserves linear, algebraic and numerical equivalence; in particular, it is compatible with the pull-back transformation for Picard functors.

3.1.2 Push-forward

A finite dominant morphism $f : Y \to X$ induces push-forward morphisms $\mathbf{f}_* : \mathbf{Div}_{Y/k} \to \mathbf{Div}_{X/k}$ and $\mathbf{f}_* : \mathbf{Pic}_{Y/k} \to \mathbf{Pic}_{X/k}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Div}_{Y/k} & \xrightarrow{\mathbf{f}_*} & \mathbf{Div}_{X/k} \\ {\scriptstyle \mathbf{A}_{Y/k}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{A}_{X/k}} \\ \mathbf{Pic}_{Y/k} & \xrightarrow{\mathbf{f}_*} & \mathbf{Pic}_{X/k} \end{array}$$

In order to verify this, we have to define the push-forward maps on the corresponding functors. Actually, we will only define the push-forward maps $f_* : \mathrm{Div}_{Y/k}(T) \to \mathrm{Div}_{X/k}(T)$ and $f_* : \mathrm{Pic}(Y_T) \to \mathrm{Pic}(X_T)$ for any T over k; the functoriality and the commuting of the diagram will be obvious from the definition. This will be done in a way very similar to [EGAII, 6.5.5] and [EGAIV-4, 21.5.3], but since we are using slighty different conditions, we will include the construction for the convenience of the reader.

First we need a small lemma:

Lemma 3.1.1. *Let A be a normal domain over a field k, K its ring of fractions, L a finite field extension of K and B the normalization of A in L. If R is an arbitrary k-algebra, then $L_R := L \otimes_k R$ is a finite free module over $K_R := K \otimes_k R$; we define a norm map $N_{L_R|K_R} : L_R \to K_R$, $\lambda \mapsto \det(m_\lambda)$, where m_λ is the endomorphism of the free K_R-module L_R given by multiplication with λ and $\det(m_\lambda)$ is its determinant. The restriction of $N_{L_R|K_R}$ to $B_R := B \otimes_k R \subseteq L_R$ maps to $A_R := A \otimes_k R \subset K_R$; we call it $N_{B_R|A_R}$. Furthermore, $\beta \in B_R$ is regular if and only if $N_{B_R|A_R}(\beta)$ is regular in A_R.*

Proof. The first statements are obvious. To show that the restriction of $N_{L_R|K_R}$ to B_R maps to A_R, let M be the normal closure of $L|K$ and C the normalization of A in M. Let $\sigma_1 = \mathrm{id}_L, \ldots, \sigma_r$ be the distinct embeddings of $L|K$ into $M|K$, and let $s = [L\colon K]_i$ be the inseparable degree of the extension $L|K$. Then $N_{L_R|K_R}(\lambda) = (\prod_{i=1}^r \sigma_i\lambda)^s$ by [La02, proposition IV.5.6.][1], where we define $\sigma_i : L_R \to M_R := M \otimes_k R$ by $\sigma_i(\sum_j l_j \otimes r_j) = \sum_j (\sigma_i l_j) \otimes r_j$. Since C is integrally closed, it follows that for $\beta \in B_R$, $N_{L_R|K_R}(\beta)$ lies in both K_R (by construction) and $C_R := C \otimes_k R$ (since all $\sigma_i\beta$ lie in C_R), hence $N_{L_R|K_R}(\beta) \in C_R \cap K_R = A_R$.

In order to show the regularity statement, it is enough to prove that $\lambda \in L_R$ is regular if and only if $N_{L_R|K_R}(\lambda) = \det(m_\lambda) \in K_R$ is regular. For this, we cite [EGAIV-4, 21.5.2]. □

Coming back to our situation, we define a map of sheaves $N_{f_{T*}\mathcal{O}_{Y_T}|\mathcal{O}_{X_T}} : f_*\mathcal{O}_{Y_T} \to \mathcal{O}_{X_T}$ by virtue of the preceding lemma (glueing the local data). It is clear from the definition that this morphism is multiplicative and sends $1_{f_*\mathcal{O}_{Y_T}}$ to $1_{\mathcal{O}_{X_T}}$, hence gives a morphism of sheaves of multiplicative groups $N_{f_{T*}\mathcal{O}_{Y_T}|\mathcal{O}_{X_T}} : (f_*\mathcal{O}_{Y_T})^\times \to \mathcal{O}_{X_T}^\times$.

Given an invertible sheaf $\mathcal{M}$ on Y_T, $f_{T*}\mathcal{M}$ is locally free over $f_{T*}\mathcal{O}_{Y_T}$ of rank one by [EGAII, 6.1.12], so we can find a set of pairs $\{(U_\lambda, \eta_\lambda)\}_\lambda$, where $\mathfrak{U} = \{U_\lambda\}_\lambda$ is an open cover of X_T and the $\eta_\lambda : (f_{T*}\mathcal{M})|_{U_\lambda} \xrightarrow{\sim} (f_{T*}\mathcal{O}_{Y_T})|_{U_\lambda}$ are isomorphisms. For arbitrary λ, μ, the automorphism $\eta_\lambda|_{U_\lambda \cap U_\mu} \circ \eta_\lambda|^{-1}_{U_\lambda \cap U_\mu}$ of $(f_{T*}\mathcal{O}_{Y_T})|_{U_\lambda \cap U_\mu}$ can be canonically identified with an element $\omega_{\lambda\mu} \in (f_{T*}\mathcal{O}_{Y_T})(U_\lambda \cap U_\mu)^\times$, giving a 1-cocycle $(\omega_{\lambda\mu})$ of $\mathfrak{U}$ with values in $(f_{T*}\mathcal{O}_{Y_T})^\times$. By the properties of the norm, $(N_{f_{T*}\mathcal{O}_{Y_T}|\mathcal{O}_{X_T}}(\omega_{\lambda\mu}))$ is a 1-cocycle of $\mathfrak{U}$ with values in $\mathcal{O}_{X_T}^\times$, corresponding (uniquely up to isomorphism) to an invertible sheaf $\mathcal{L} =: N_{Y_T|X_T}(\mathcal{M})$ of X_T.

This defines a map $N_{Y_T|X_T} : \mathrm{Pic}(Y_T) \to \mathrm{Pic}(X_T)$. By looking at the properties of the norm, one can easily deduce that this actually is a group homomorphism and that furthermore the projection formula $N_{Y_T|X_T}((f_T)^*\mathcal{L}) = \mathcal{L}^{\otimes n}$ holds. In particular, we get a group homomorphism

$$N_{Y|X}(T) : \mathrm{Pic}_{Y/k}(T) \to \mathrm{Pic}_{X/k}(T),$$

which finally yields a morphism $\mathbf{f}_* := \mathbf{N}_{Y|X} : \mathbf{Pic}_{Y/k} \to \mathbf{Pic}_{X/k}$.

Now consider a relative effective Cartier divisor $E \in \mathrm{Div}_{Y/k}(T)$. Then E is given by a collection of (V_i, s_i), where $\mathfrak{V} = (V_i)_i$ is an open covering of Y_T, $s_i \in \mathcal{O}_{Y_T}(V_i)$, such that the following properties hold:

- $s_j|_{V_i \cap V_j} \in s_i|_{V_i \cap V_j}\mathcal{O}_{Y_T}(V_i \cap V_j)^\times$ for every i, j,
- for all i, s_i is regular on $Y_{\kappa(t)}$ for any $t \in \mathrm{pr}_2(U_i) \subset T$.

Since f is affine, we can assume the V_i to be of the form $V_i = f_T^{-1}(U_i)$, where $\mathfrak{U} = (U_i)_i$ is an open covering of X_T; this allows us to consider s_i as an element of $(f_{T*}\mathcal{O}_{Y_T})(U_i)$. Thus we can define $r_i := N_{f_{T*}\mathcal{O}_{Y_T}|\mathcal{O}_{X_T}}(s_i) \in \mathcal{O}_{X_T}(U_i)\ \forall\ i$. The multiplicativity of the norm immediately implies $r_j|_{U_i \cap U_j} \in r_i|_{U_i \cap U_j}\mathcal{O}_{X_T}(U_i \cap U_j)^\times$ for all i, j, and by the last part of the lemma above, r_i is regular on $Y_{\kappa(t)}$ for any

[1] The cited proposition only claims it in the case where $R = k$, but the proof carries over to general R.

i and any $t \in \mathrm{pr}_2(U_i)$. Therefore, the collection (U_i, r_i) defines a relative effective divisor, which we will denote by f_*E.

So to every $E \in \mathrm{Div}_{Y/k}(T)$, we can assign a $D := f_*E \in \mathrm{Div}_{X/k}(T)$; from the properties of the norm map one can easily deduce that this assignment is well-defined, gives a homomorphism of monoids and is functorial. We therefore get a morphism

$$\mathbf{f}_* : \mathbf{Div}_{Y/k} \to \mathbf{Div}_{X/k}\,.$$

The push-forward map f_* on divisors preserves linear, algebraic and numerical equivalence, in particular, the diagram in the beginning of this subsection commutes.

Remark 3.1.2. For a finite dominant morphism $f : Y \to X$ of normal varieties over a field k, one can define a push-forward map $f_* : Z^1(Y) \to Z^1(X)$ on Weil divisors by setting $f_*W := [k(W) : k(f(W))]f(W)$ for a Weil prime divisor W on Y and extending linearly ([Fu98, section 1.4]). It is easy to see that the restriction of this map to the Cartier divisors on Y is just the map defined above.

3.1.3 Properties

As mentioned earlier, we have $N_{Y_T|X_T}((f_T)^*\mathcal{L}) = \mathcal{L}^{\otimes n}$ for any $\mathcal{L} \in \mathrm{Pic}_{X/k}(T)$, where $n = \deg f$. Similarly, we obviously have $f_*(f^*D) = nD$ for any $D \in \mathrm{Div}_{X/k}(T)$. Therefore, the maps $\mathbf{f}_* \circ \mathbf{f}^* : \mathbf{Pic}_{X/k} \to \mathbf{Pic}_{X/k}$ and $\mathbf{f}_* \circ \mathbf{f}^* : \mathbf{Div}_{X/k} \to \mathbf{Div}_{X/k}$ are just the nth power maps on these monoid schemes.

The composition $f^* \circ f_* : \mathrm{Div}_{Y/k} \to \mathrm{Div}_{Y/k}$ is a little more complicated. If $f : Y \to X$ is a Galois cover with Galois group G, then G also acts on $\mathrm{Div}_{Y/k}$ per natural transformations, and $(f^* \circ f_*)(E) = \sum_{\sigma\in G} \sigma E$ for $E \in \mathrm{Div}_{Y/k}(T)$, as can be derived from the proof of lemma 3.1.1. In the general case, we can still derive that $(f^* \circ f_*)(E) - E \in \mathrm{Div}_{Y/k}(T)$, i.e. is effective.

Proposition 3.1.3. *The morphism $\mathbf{f}_* : \mathbf{Div}_{Y/k} \to \mathbf{Div}_{X/k}$ is proper in the local sense: For any open subscheme $\mathbf{V}$ of $\mathbf{Div}_{X/k}$ having only finitely many connected components, the restricted map $\mathbf{f}_*^{-1}\mathbf{V} \to \mathbf{V}$ is proper.*

Proof. We can assume k to be algebraically closed. It will be enough to consider the case $\mathbf{V} = \mathbf{Div}_{X/k}^{[D]_{num}}$ of $\mathbf{Div}_{X/k}$; then $\mathbf{f}_*^{-1}(\mathbf{V})$ is the disjoint union of $\mathbf{Div}_{Y/k}^{[E]_{num}}$ for all classes $[E]_{num}$ which map to $[D]_{num}$ under f_*. But this union is only a finite union: Using the notation of corollary 2.4.2, $[E]_{num} \leq [f^*D]_{num}$ for every such $[E]_{num}$, and there are only finitely many such classes. Now using the fact that all these $\mathbf{Div}_{Y/k}^{[E]_{num}}$ are of finite type over k, we immediately get that the map is of finite type.

All that is left to show is that for $[E]_{num} \in N^1(Y)$, $[D]_{num} \in N^1(X)$ with $f_*[E]_{num} = [D]_{num}$, the restricted morphism $\mathbf{f}_* : \mathbf{Div}_{Y/k}^{[E]_{num}} \to \mathbf{Div}_{X/k}^{[D]_{num}}$ is proper. But both the domain and the target are proper over k, so this follows from [Ha77, II.4.8 (e)]. $\square$

3.2 More definitions and notations

3.2.1 Galois closure for separable covers

The only purpose of this short section is to describe a context and fix a notation that will be used at several parts of this thesis.

Let $f : Y \to X$ be a separable cover of normal varieties over k. Let $L|k(X)$ be the Galois closure of $k(Y)|k(X)$, and let Z be the normalization of X in L. Set $G = \mathrm{Gal}(L|k(X)$, then G acts on Z, and $Z/G \cong X$. Similarly, for $H = \mathrm{Gal}(L|k(Y)$ we have $Z/H \cong Y$. Define $g : Z \to X$ and $h : Z \to Y$ to be the corresponding quotient morphisms.

3.2.2 Split and c-split divisors

Assume $f : Y \to X$ is a finite cover of normal geometrically integral projective varieties over a field k.

Definition 3.2.1. Let D be a relative effective Cartier divisor on X_T/T.

1. We say that D is *c-split* if there exists a relative effective Cartier divisor E on Y_T/T such that $D = f_*E$. If there are $n = \deg(f)$ different such E_i and $E_1 + \ldots + E_n = f^*D$, we say D is *completely c-split*.
2. If T is the spectrum of a field, we say that D is *split* if there is an effective Weil divisor E_W on Y_T such that f_*E_W is the Weil divisor associated to D. Analogously, define *completely split* divisors.

Remark 3.2.2. a) If D splits, D is a prime divisor if and only if E_W is a Weil prime divisor and $k(E_W) = k(D)$. In this case, the notion of complete splitting above coincides with the usual notion of complete splitting in Hilbert decomposition theory.

b) In this context, assume E is a Cartier prime divisor on Y that does not lie in the branch locus. Then $f : Y \to X$ factors through a unique intermediate cover $Y \xrightarrow{f''} Y' \xrightarrow{f'} X$ such that for $E'_W = f''(E)$, we have $f''_*E = \deg(f'')E'_W$ and $f'_*E'_W = f(E)$ – in other words, such that $[k(Y) : k(Y')] = [k(E) : k(f''(E))] = [k(E) : k(f(E))]$.

This is trivial if $k(Y)|k(X)$ is inseparable, since then it is ramified everywhere. So we can assume that $k(Y)|k(X)$ is separable. Using the notation of section 3.2.1, choose a Weil prime divisor F_W on Z_T such that $h(F_W) = E$ and set $Y' = Z/H'$ with $H' = HG_{F_W}$. Then $Y \to X$ factors through Y', and we have

$$[k(E) : k(f''(E))] = [(G_{F_W} \cap H') : (G_{F_W} \cap H)] = [HG_{F_W} : H] = [k(Y) : k(Y')]$$

and $[k(E) : k(f(E))] = [(G_{F_W} \cap G) : (G_{F_W} \cap H)] = [HG_{F_W} : H] = [k(Y) : k(Y')]$. The uniqueness is clear from the construction.

c) Still in the same context, assume E is a Cartier prime divisor on Y outside branch locus such that $f_*E = \deg(f) \cdot f(E)$, or in other words, such that $[k(E) : k(f(E))] = [k(Y) : k(X)]$. Again, construct the 'Galois closure' $Z \to X$ as in section 3.2.1, using the same notation. Then h^*E is G-invariant.

   Indeed, if we split $h^*E = F_{W,1} + \ldots + F_{W,r}$ into a sum of distinct Weil prime divisors $F_{W,i}$, $i = 1, \ldots, r$, then necessarily $g(F_{W.i}) = f(E)\ \forall\ i$, so all $F_{W,i}$ are conjugated under G. On the other hand, the above condition on E implies that $[G : H] = \deg(f) = [G_{F_{W,1}} : H_{F_{W,1}}] = [G_{F_{W,1}} : G_{F_{W,1}} \cap H] = [HG_{F_{W,1}} : H]$, so $HG_{F_{W,1}} = G$. Therefore, the number of different conjugates of $F_{W,1}$ under G is $[G : G_{F_{W,1}}] = [H : H_{F_{W,1}}] = r$. Thus, the action of G just permutes the $F_{W,i}$, hence fixes h^*E.

Proposition/Definition 3.2.3. Let D be a Cartier divisor on X. Define a subfunctor $\widetilde{Sc}_{D/(Y\to X)/k}$ of $\mathrm{LinSys}_{D/X/k}$ by

$$\widetilde{Sc}_{D/(Y\to X)/k}(T) := \left\{D' \in \mathrm{LinSys}_{D/X/k}(T) | D' \text{ is c-split in } Y_T\right\}.$$

$\widetilde{Sc}_{D/(Y\to X)/k}$ is representable by the closed subscheme

$$\widetilde{\mathbf{Sc}}_{D,Y} = \widetilde{\mathbf{Sc}}_{D/(Y\to X)/k} := \mathbf{L}_{D/X/k} \cap \mathbf{f}_* \, \mathbf{Div_{Y/k}}$$

of $\mathbf{L}_D$. (We will use the simpler notation unless there is a possible ambiguity about what the base scheme might be.) We denote its (finite) dimension by $\tilde{s}_c^Y(D) = \tilde{s}_c(D)$.

We can find an easy upper bound for $\tilde{s}_c(D)$:

Lemma 3.2.4. $\tilde{s}_c(D) \leq \max\{\dim \mathbf{Div}_{Y/k}^{[E]_{num}} \mid f_*[E]_{num} = [D]_{num}\}$.

Proof. This is clear, since $\widetilde{\mathbf{Sc}}_{D,Y}$ lies in the image of $\bigcup_{f_*[E]_{num}=[D]_{num}} \mathbf{Div}_{Y/k}^{[E]_{num}}$ under $\mathbf{f}_*$. □

3.3 Behavior of volume in finite covers

Let k be a field, X be a projective variety over k of dimension d, D be a Cartier divisor on X. Then the following generalization of the one-dimensional Riemann-Roch theorem holds:

Proposition 3.3.1 (asymptotic Riemann-Roch)**.**

a) We have $h^0(X, mD) = O(m^d)$, or more generally $h^0(X, \mathcal{F}(mD)) = O(m^d)$ for any coherent sheaf $\mathcal{F}$ on X.

b) If D is nef, we have

$$h^0(X, mD) = \frac{\int_X D^d}{d!} \cdot m^d + O(m^{d-1}).$$

Proof. [De01, 1.31] □

This proposition suggests the following

Definition 3.3.2. For any Cartier divisor D on X, we define its *volume* to be

$$\mathrm{vol}(D) = \mathrm{vol}_X(D) = \limsup_{m\to\infty} \frac{h^0(X, mD)}{m^d/d!}.$$

Proposition 3.3.3. *a) For every nef divisor D, we have* $\mathrm{vol}(D) = \int_X D^d$.

b) For $a \in \mathbb{N}$, we have $\mathrm{vol}(aD) = a^d \,\mathrm{vol}(D)$.

c) A Cartier divisor D is big if and only if $\mathrm{vol}(D) > 0$.

d) The volume increases in effective directions, i.e. if $D, E \in$ $\mathrm{Div}(X)$ *and E is effective, then* $\mathrm{vol}(D + E) \geq \mathrm{vol}(D)$.

Part b) of the last proposition allows one to extend the notion of the volume in a unique way to $\mathbb{Q}$-Cartier divisors, so that all these properties still hold on the $\mathrm{Div}(X)_{\mathbb{Q}}$.

Proposition 3.3.4. *Let D be a big divisor on X. Then for all integers $m \gg 0$, the map $\mathbf{Div}_{X/k}^{[mD]_{num}} \to \mathbf{Pic}_{X/k}^{[mD]_{num}}$ is surjective.*

Proof. Without loss of generality , we can assume k to algebraically closed. It is enough to prove that for $m \gg 0$, we have

$$h^0(X, mD + N) > 0 \;\forall\; N \equiv 0.$$

Actually, if we prove this for one divisor D, we prove it for every divisor of the form $D + E$, where E is an effective divisor. Therefore, we can restrict ourselves to only consider ample divisors D. In this case, on the one hand we have Fujita's vanishing theorem [La04a, 1.4.35, 1.4.36], which tells us that there is an $m(D)$ such that for all $m > m(D)$, $h^i(X, mD + N) > 0 \;\forall\; N \equiv 0, i > 0$, and on the other hand $\chi(mD + N) = \chi(mD)$ for any numerically trivial N. Taking these two facts together, we get for $m > m(D)$:

$$h^0(X, mD + N) = \chi(X, mD + N) = \chi(X, mD) = h^0(X, mD),$$

from which the assertion follows by the ampleness of D. □

Corollary 3.3.5. *The volume only depends on the numerical equivalence class.*

Proof. Let D, D' be two numerically equivalent divisors. If D is not big, so is D', and both $\mathrm{vol}(D)$ and $\mathrm{vol}(D')$ are 0 by part c) of proposition 3.3.3. Hence, we can assume that D (and hence also D') is big. By proposition 3.3.3 b), it is enough to prove $\mathrm{vol}(aD) = \mathrm{vol}(aD')$ for some positive integer a. Replacing D and D' by aD and aD' if necessary, we can assume that $h^0(X, D+N) > 0$ for any $N \equiv 0$ by proposition 3.3.4. In particular, this is true for $N = m(D' - D)$, $m \in \mathbb{N}$. Thus $h^0(X, mD')) = h^0(X, (m-1)D + (D + m(D' - D))) \geq h^0(X, (m-1)D)$. Dividing by $\frac{m^d}{d!}$ and taking the limes superior on both sides then yields $\mathrm{vol}(D') \geq \mathrm{vol}(D)$. By symmetry, we also have $\mathrm{vol}(D) \geq \mathrm{vol}(D')$, so we get the desired equality. □

Now we investigate the behavior of volume in finite covers.

Lemma 3.3.6. *Let $f : Y \to X$ be a proper, dominant, generically finite morphism of projective varieties over k. For any $D \in \mathrm{Div}(X)$, we have*

$$\mathrm{vol}_Y(f^*D) = \deg(f)\,\mathrm{vol}_X(D).$$

Proof. By the projection formula,

$$H^0(Y, \mathcal{O}_Y(mf^*D)) = H^0(X, f_*(\mathcal{O}_Y(mf^*D))) \cong H^0(X, (f_*\mathcal{O}_Y)(mD)),$$

so we restrict our attention to $f_*\mathcal{O}_Y$. There is an open dense subset U of X such that $f_*\mathcal{O}_Y$ is free of rank $n = \deg(f)$, so $(f_*\mathcal{O}_Y)|_U \simeq \mathcal{O}_U^n$. This isomorphism gives an injection $f_*\mathcal{O}_Y \hookrightarrow \mathcal{K}_X^n$, where $\mathcal{K}_X$ is the sheaf of total quotients rings of $\mathcal{O}_X$. Let $\mathcal{G} = f_*\mathcal{O}_Y \cap \mathcal{O}_X^n$ and define $\mathcal{G}_1$ and $\mathcal{G}_2$ by the exact sequences of sheaves

$$\begin{aligned} 0 \to \mathcal{G} \to f_*\mathcal{O}_Y \to \mathcal{G}_1 \to 0, \\ 0 \to \mathcal{G} \to \quad \mathcal{O}_X^n \quad \to \mathcal{G}_2 \to 0. \end{aligned}$$

The supports of $\mathcal{G}_1$ and $\mathcal{G}_2$ do not meet U, hence have dimension less than d. Using proposition 3.3.1 a) and the long exact sequence of cohomology, this implies

$$\begin{aligned} h^0(Y, \mathcal{O}_Y(mf^*D)) = h^0(X, (f_*\mathcal{O}_Y)(mD)) &= h^0(X, \mathcal{O}_X^n(mD)) + O(m^{d-1}) \\ &= n \cdot h^0(X, \mathcal{O}_X(mD)) + O(m^{d-1}), \end{aligned}$$

from which the assertion follows. □

Proposition 3.3.7 (log-concavity of the volume)**.** *If D, D' are big divisors on X, then*

$$\mathrm{vol}(D)^{\frac{1}{d}} + \mathrm{vol}(D')^{\frac{1}{d}} \leq \mathrm{vol}(D + D')^{\frac{1}{d}}.$$

Proof. The proof is the same as the one in [La04b, 11.4.9], but we include it for completeness, since there it is only claimed in characteristic zero.

For ample divisors $D_1, \ldots, D_d$, we have the generalized inequality of Hodge type

$$\Big(\int_X D_1 \cdots D_d\Big)^d \geq \Big(\int_X D_1^d\Big) \cdots \Big(\int_X D_d^d\Big),$$

which follows easily from the 2-dimensional Hodge inequality ([La04a, 1.6.1, 1.6.5]). If D and D' are ample divisors, this implies in particular

$$\begin{aligned}\big(\mathrm{vol}(D)^{\frac{1}{d}} + \mathrm{vol}(D')^{\frac{1}{d}}\big)^d &= \Big[\Big(\int_X D^d\Big)^{\frac{1}{d}} + \Big(\int_X D'^d\Big)^{\frac{1}{d}}\Big]^d \\ &= \sum_{j=0}^d \binom{d}{j}\Big(\int_X D^d\Big)^{\frac{j}{d}}\Big(\int_X D'^d\Big)^{\frac{d-j}{d}} \\ &\leq \sum_{j=0}^d \binom{d}{j}\Big(\int_X D^j \cdot D'^{d-j}\Big) = \int_X (D+D')^d = \mathrm{vol}(D+D'),\end{aligned}$$

so the assertion holds in the case of ample divisors and, for that matter, also for ample $\mathbb{Q}$-divisors. In the general case, we use Fujita's approximation theorem, originally proven in the characteristic zero case in [Fuj94] and extended to any characteristic in [Ta07]:

Given any big divisor D on X and every $\varepsilon > 0$, there exists a modification $\mu : X' \to X$ and a decomposition $\pi^* D = A + E$ in $\mathrm{Div}(X')_{\mathbb{Q}}$, where A is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor, such that $\mathrm{vol}_{X'}(A) > \mathrm{vol}_X(D) - \varepsilon$.

Now in our situation, we fix $\varepsilon > 0$ and construct a simultaneous Fujita approximation

$$\mu : X' \to X, \quad \mu^* D = A + E, \quad \mu^* D' = A' + E',$$

with $\mathrm{vol}_{X'}(A)^{\frac{1}{d}} > \mathrm{vol}_X(D)^{\frac{1}{d}} - \frac{\varepsilon}{2}$ and $\mathrm{vol}_{X'}(A')^{\frac{1}{d}} > \mathrm{vol}_X(D')^{\frac{1}{d}} - \frac{\varepsilon}{2}$. Then since $\mu^*(D+D') - (A+A')$ is effective, we have

$$\begin{aligned}\mathrm{vol}_X(D+D')^{\frac{1}{d}} &= \mathrm{vol}_{X'}(\mu^*(D+D'))^{\frac{1}{d}} \\ &\geq \mathrm{vol}_{X'}(A+A')^{\frac{1}{d}} \\ &\geq \mathrm{vol}_{X'}(A)^{\frac{1}{d}} + \mathrm{vol}_{X'}(A')^{\frac{1}{d}} \\ &\geq \mathrm{vol}_X(D)^{\frac{1}{d}} + \mathrm{vol}_X(D')^{\frac{1}{d}} - \varepsilon.\end{aligned}$$

As $\epsilon \searrow 0$, the proposition follows. □

Proposition 3.3.8. *Let k be a field, $f : Y \to X$ be a finite, dominant morphism of normal projective varieties over k of dimension d, and let $E \in \mathrm{Div}(Y)$, $D \in \mathrm{Div}(X)$ such that $f_* E = D$. Then*

$$\mathrm{vol}_Y(E) \leq \frac{1}{\deg(f)^{d-1}} \mathrm{vol}_X(D).$$

Proof. If $k(Y)|k(X)$ is separable, use the notation of section 3.2.1. Then we have

$$\sum_{\bar{\sigma}\in G/H} \sigma h^* E = g^* D \in \mathrm{Div}(Z),$$

since both divisors are G-invariant and have the same image under $g_* = f_* h_*$:

$$\begin{aligned} g_* \textstyle\sum_{\bar{\sigma}\in G/H} \sigma h^* E &= (G:H)\cdot g_* h^* E = (G:H)\cdot f_* h_* h^* E \\ &= (G:H)\cdot \mathrm{ord}(H)\cdot f_* E \\ &= \mathrm{ord}(G)\cdot D = g_* g^* D \end{aligned}$$

(here we use the fact that $\mathrm{Div}(X)_{\mathbb{Q}} = (\mathrm{Div}(Z)_{\mathbb{Q}})^G$, compare e.g. with [Fu98, 1.7.6]). We obviously have $\mathrm{vol}_Z(h^*E) = \mathrm{vol}_Z(\sigma h^* E)\ \forall\ \sigma \in G$, so by propositions 3.3.6 and 3.3.7 we get

$$\begin{aligned} (\mathrm{ord}(G)\,\mathrm{vol}_X(D))^{\frac{1}{d}} = \mathrm{vol}_Z(g^*D)^{\frac{1}{d}} &= \mathrm{vol}_Z\Big(\sum_{\bar{\sigma}\in G/H} \sigma h^* E\Big)^{\frac{1}{d}} \\ &\geq \sum_{\bar{\sigma}\in G/H} \mathrm{vol}_Z\Big(\sigma h^* E\Big)^{\frac{1}{d}} = (G:H)\,\mathrm{vol}_Z(h^*E)^{\frac{1}{d}} \\ &= (G:H)\cdot(\mathrm{ord}(H)\,\mathrm{vol}_Y(E))^{\frac{1}{d}} \end{aligned}$$

Taking dth powers yields $\mathrm{vol}_X(D) \geq (G:H)^{d-1}\,\mathrm{vol}_Y(E) = (\deg f)^{d-1}\,\mathrm{vol}_Y(E)$.

In the case that $k(Y)|k(X)$ is purely inseparable, the norm $N_{k(Y)|k(X)}$ just raises every element of $k(Y)$ to the qth power, where $q = \deg(f)$; therefore

$$\begin{aligned} \mathrm{vol}_X(D) &= \frac{1}{\deg(f)}\,\mathrm{vol}_Y(f^* f_* E) = \frac{1}{\deg(f)}\,\mathrm{vol}_Y(qE) = \frac{q^d}{\deg(f)}\,\mathrm{vol}_Y(E) \\ &= (\deg f)^{d-1}\,\mathrm{vol}_Y(E). \end{aligned}$$

In the general case, we can split $f : Y \to X$ into finite covers $Y \to X' \to X$, where X' is a normal projective variety, $k(Y)|k(X')$ is purely inseparable and $k(X')|k(X)$ is separable. The assertion then follows by composition. □

Actually, in the following we will need a seemingly stronger, but in fact equivalent result. For this, let us first fix some

Notation. Let $D \in \mathrm{Div}(X)$ be any Cartier divisor that is not numerically trivial. We define

$$\mathrm{Div}_D(Y) = \{E \in \mathrm{Div}(Y) | f_* E \sim mD \text{ for some } m \in \mathbb{Z}\} \subset \mathrm{Div}(Y),$$

$$\Lambda_D = N^1_D(Y) = \{\eta \in N^1(Y) | f_*\eta \in \mathbb{Z}\cdot[D]_{num}\} \subset N^1(Y).$$

$N^1_D(Y)_\mathbb{R}$ is a subspace of $N^1(Y)_\mathbb{R}$, containing Λ_D as a complete lattice.

For $E \in \mathrm{Div}_D(Y)$, define its degree $\deg(E) \in \mathbb{Z}$ by $f_*E =: \deg(E)D$. This induces maps $\deg : N^1_D(Y) \to \mathbb{Z}$ and $\deg : N^1_D(Y)_\mathbb{R} \to \mathbb{R}$. Let H_D be the affine hyperplane $\{\eta \in N^1_D(Y)_\mathbb{R} | \deg \eta = 1\}$ in $N^1_D(Y)_\mathbb{R}$. Then the intersection of H_D with the pseudoeffective cone $\overline{\mathrm{Eff}}(Y)$ in $N^1(Y)_\mathbb{R}$ is compact, since it is a closed subset of the set $\{\eta \in \overline{\mathrm{Eff}}(Y) | f^*[D]_{num} - \eta \in \overline{\mathrm{Eff}}(Y)\}$, which is compact by lemma 2.4.1.

Proposition 3.3.9. *In the context of proposition 3.3.8, let $D \in \mathrm{Div}(X)$ be a big divisor. Then*

$$\limsup_{m\to\infty} \frac{\max\{h^0(Y,E) | E \in \mathrm{Div}(Y), f_*E \equiv mD\}}{m^d/d!} \le \frac{1}{\deg(f)^{d-1}} \mathrm{vol}(D).$$

Proof. Set $C := \frac{1}{\deg(f)^{d-1}} \mathrm{vol}(D)$. Assume there exist an $\varepsilon > 0$ and divisors E_i on Y with $[E_i]_{num} \in N^1_D(Y)$ of degree m_i, $m_i \to \infty$, such that

$$\frac{h^0(Y,E_i)}{m_i^d/d!} \ge C(1+\varepsilon) \quad \forall\, i.$$

Since $P = H_D \cap \overline{\mathrm{Eff}}(Y)$ is compact, after changing to a subsequence, we can assume that the $\eta_i = \frac{1}{m_i}[E_i]_{num}$ converge in P. In fact, after replacing ε by a smaller positive number ε' and the E_i by $E_i' = E_i + \lfloor \frac{m_i}{n}\alpha \rfloor f^*D$ with $1 < 1+\alpha < \sqrt[d]{\frac{1+\varepsilon}{1+\varepsilon'}}$, we can even assume that the η_i converge to a point in the interior of P, i.e. to a big class $\eta \in H_D$. We will show that in a small neighborhood of η, every rational divisor class has volume greater than given by lemma 3.3.8.

To achieve this, choose big rational divisors classes $\beta_0, \ldots, \beta_s \in H_D$ such that the β_i form a basis of $N^1_D(X)$ and η lies in the interior of the simplex $S_\beta \subset H_D$ with vertices β_i. There exists an $l \gg 0$ such that for every $i = 0, \ldots, s$, $l\beta_i \in \Lambda_D$ and every linear equivalence class mapping to $l\beta_i$ is effective (see 3.3.4). Let Λ_B denote the sublattice of Λ_D generated by the $l\beta_i \in \Lambda_D$; since the β_i form a basis of $N^1_D(Y)$, there is a positive integer L such that $L\Lambda_D \subseteq \Lambda_B$.

After changing to a subsequence, we can assume that all η_i lie inside S_β and all $[E_i]_{num} = m_i\eta_i \in \Lambda_D$ have the same coset class in Λ_D/Λ_B. Write $\eta_i = \sum_{j=0}^s \mu_{ij}\beta_j$ with $\mu_{ij} > 0$ and $\sum_{j=0}^s \mu_{ij} = 1\ \forall\ i$. For $\delta > 0$ small enough, choose a big divisor $\tilde{E} \in \mathrm{Div}_D(Y)$ of degree $\tilde{m}$ such that $[\tilde{E}]_{num}$ has the same class in Λ_D/Λ_B as the $[E_i]_{num}$ and such that for $\frac{1}{\tilde{m}}[\tilde{E}]_{num} = \tilde{\eta} = \sum_{j=0}^s \tilde{\mu}_j\beta_j$, one has $|\mu_{ij} - \tilde{\mu}_j| < \delta^2$ for all j and all $i \gg 0$, $|\tilde{\mu}_j| \ge \delta$ for all j (in fact, one can choose $\tilde{E}$ to be one of the E_i for i big enough).

For a fixed i, we look for a p_i such that

$$E_i' = (1+p_iL)\tilde{E} - E_i = \sum_{j=0}^s ((1+p_iL)\,\tilde{m}\tilde{\mu}_j - m_i\mu_{ij})\beta_j$$

is linearly equivalent to an effective divisor. First of all, since $[E_i']_{num} \in \Lambda_B$, we have $(1+p_iL)\,\tilde{m}\tilde{\mu}_j - m_i\mu_{ij} \in \mathbb{Z}l\ \forall\ j$. For any $n_0, \ldots, n_s \in \mathbb{N}$ with at least one $n_j > 0$, any linear equivalence class mapping to $\sum_{j=0}^{s} n_j l \beta_j$ contains an effective divisor by our condition on l, so a sufficient condition on p_i is

$$(1+p_iL)\,\tilde{m}\tilde{\mu}_j - m_i\mu_{ij} > 0 \quad \forall\ j$$

Since $\frac{m_i\mu_{ij}}{\tilde{m}\tilde{\mu}_j} < \frac{m_i}{\tilde{m}}(1+\frac{\delta^2}{\tilde{\mu}_j}) \leq \frac{m_i}{\tilde{m}}(1+\delta)$, this is fulfilled for any $p_i \in \mathbb{N}$ such that $1+p_iL \geq \frac{m_i}{\tilde{m}}(1+\delta)$. If we take p_i to be the smallest such integer, then we have $1+p_iL < \frac{m_i}{\tilde{m}}(1+\delta) + L < \frac{m_i}{\tilde{m}}(1+2\delta)$ for $i \gg 0$.

Because $(1+p_iL)\tilde{E} - E_i$ is linear equivalent to an effective divisor, we have

$$h^0(Y,(1+p_iL)\tilde{E}) \geq h^0(Y,E_i) > C(1+\varepsilon)\frac{m_i^d}{d!} > C\frac{1+\varepsilon}{(1+2\delta)^d}\frac{\tilde{m}^d}{d!}(1+p_iL)^d$$

for $i \gg 0$. As $i \to \infty$, we get $\mathrm{vol}(\tilde{E}) \geq C\frac{1+\varepsilon}{(1+2\delta)^d}\tilde{m}^d > C\tilde{m}^d$ for δ small enough, in contradiction to proposition 3.3.8 (and 3.3.3 b)). □

Corollary 3.3.10. *In the above context, let D be a big divisor and D' be an effective divisor on X. Then*

$$\limsup_{m\to\infty} \frac{\max\{h^0(Y,E) | E \in \mathrm{Div}(Y), f_*E \equiv mD + D'\}}{m^d/d!} \leq \frac{1}{\deg(f)^{d-1}}\,\mathrm{vol}(D).$$

Proof. Set $h(\tilde{D}) := \max\{h^0(Y,E) | E \in \mathrm{Div}(Y), f_*E \equiv \tilde{D}\}$ and let $n := \deg(f)$. Since $f_*f^*\tilde{D}' = n\tilde{D}'$ for any divisor $\tilde{D}'$ on X, we have $h(\tilde{D}) \leq h(\tilde{D} + n\tilde{D}')$ for every effective divisor $\tilde{D}'$.

Fix an integer $l \in \{0, \ldots, n-1\}$. For our assertion, it will be enough to prove that for any choice of l, we have

$$\limsup_{m\to\infty, m\equiv l \bmod n} \frac{h(mD+D')}{m^d/d!} \leq \frac{1}{n^{d-1}}\,\mathrm{vol}(D).$$

Now fix a big integer $m_0 \equiv l \mod n$. For all $m \geq m_0$, $m \equiv l \mod n$, we can find $a_m \in \mathbb{N}$, $b_m \in \{0, 1, \ldots, m_0 - 1\}$, such that $m - m_0 = (a_m m_0 - b_m)n$ or, in other words, $m + b_m n = (1 + a_m n)m_0$. By our observation above, we have

$$h(mD+D') \leq h\big((m+b_mn)D + (1+a_mn)D'\big) = h\big((1+a_mn)(m_0D+D')\big).$$

Thus

$$\begin{aligned}
\limsup_{m\to\infty, m\equiv l \bmod n} \frac{h(mD+D')}{m^d/d!} &\leq \limsup_{m\to\infty, m\equiv l \bmod n} \frac{h\big((1+a_m n)(m_0D+D')\big)}{m^d/d!} \\
&= \frac{1}{m_0^d}\cdot \limsup_{m\to\infty, m\equiv l \bmod n} \frac{h\big((1+a_m n)(m_0D+D')\big)}{(1+a_m n)^d/d!} \\
&\leq \frac{1}{m_0^d}\cdot \limsup_{\tilde{m}\to\infty} \frac{h\big(\tilde{m}(m_0D+D')\big)}{\tilde{m}^d/d!} \\
&\leq \frac{1}{m_0^d n^{d-1}} \operatorname{vol}(m_0D+D') \qquad \text{by proposition 3.3.9} \\
&= \frac{1}{n^{d-1}} \operatorname{vol}\Big(D+\frac{1}{m_0}D'\Big).
\end{aligned}$$

As $m_0\to\infty$, we get the desired inequality by the continuity of the volume ([La04a, 2.2.25]). □

Corollary 3.3.11. *In the above context, let D be a big divisor and D' be an effective divisor on X. For the dimension $\tilde{s}_c(mD+D')$ of the scheme $\widetilde{\mathbf{Sc}}_{mD+D',Y}$ defined in section 3.2.2, we have*

$$\limsup_{m\to\infty} \frac{\tilde{s}_c(mD+D')}{m^d/d!} \leq \frac{1}{\deg(f)^{d-1}} \operatorname{vol}(D).$$

Proof. We can assume k to be algebraically closed. By lemma 3.2.4, we have

$$\tilde{s}_c(mD+D') \leq \max\{\dim \mathbf{Div}_{Y/k}^{[E]_{num}} \mid f_*E \equiv mD+D'\}.$$

So we have to give upper bounds for the dimension of the $\mathbf{Div}_{Y/k}^{[E]_{num}}$. For fixed E, consider the Abel map $W := \mathbf{Div}_{Y/k}^{[E]_{num}} \to \mathbf{Pic}_{Y/k}^{[E]_{num}} =: V$. Then by [EGAIV-2, 5.6.7], we have

$$\dim W \leq \dim V + \max_{v\in V}\{\dim W_v\}.$$

Because of the upper semicontinuity of the dimension of the fiber ([EGAIV-3, 13.1.5]), it is actually enough to take the maximum only over the closed points of V. But the closed points correspond to invertible sheaves $\mathcal{L}$ that are numerically equivalent to $\mathcal{L}(E)$, and the corresponding fibers $V_{\mathcal{L}}$ are just the schemes $\mathbf{L}_{\mathcal{L}/X/k}$ (see section 2.2), hence have dimension $h^0(X,\mathcal{L})-1$. Therefore

$$\dim \mathbf{Div}_{Y/k}^{[E]_{num}} \leq \dim \mathbf{Pic}_{Y/k}^{\tau} - 1 + \max_{E'\equiv E}\{h^0(X,E')\}$$

and

$$\tilde{s}_c(mD+D') \leq \dim \mathbf{Pic}_{Y/k}^{\tau} - 1 + \max\{h^0(Y,E) | E\in \operatorname{Div}(Y), f_*E \equiv mD+D'\}.$$

Now the claim follows directly from the last corollary. □

3.4 The scheme representing geometrically integral Cartier divisors

To a point $z \in \mathbf{Div}_{X/k}$, we can associate an effective Cartier divisor $D(z)$ of $X_{k(z)}$ which corresponds to the unique map $\operatorname{Spec} k(z) \to \mathbf{Div}_{X/k}$ with image $\{z\}$. Consider the set

$$U = \{z \in \mathbf{Div}_{X/k} \,|D(z) \text{ is geometrically integral}\}.$$

Proposition/Definition 3.4.1. The set U is an open subset of $\mathbf{Div}_{X/k}$. We define $\mathbf{GIDiv}_{X/k}$ to be the open subscheme of $\mathbf{Div}_{X/k}$ corresponding to U; it represents the functor $\mathrm{GIDiv}_{X/k}$ defined by

$$\mathrm{GIDiv}_{X/k}(T) = \{D \in \mathrm{Div}_{X/k}(T) | D_t \text{ geometrically integral } \ \forall\, t \in T\}.$$

Proof. Let D be a relative effective divisor on X_T/T, $\varphi : T \to \mathbf{Div}_{X/k}$ be the corresponding morphism. Then we claim

$$\varphi^{-1}(U) = \{t \in T | D_t \text{ is geometrically integral }\}. \tag{3.4.1}$$

In fact, D_t corresponds to the morphism $\operatorname{Spec} k(t) \to \mathbf{Div}_{X/k}$ given by the composition of the natural morphism $\operatorname{Spec} k(t) \to T$ and φ, so by the definition of U, D_t is geometrically integral if and only if $\varphi(t) \in U$.

Now D is proper and flat over T, so by [EGAIV-3, 12.2.4 (vii)] the set of all $t \in T$ for which D_t is geometrically integral is open in T, i.e. $\varphi^{-1}(U)$ is an open subset of T. For $T = \mathbf{Div}_{X/k}$, $\varphi = \mathrm{id}_{\mathbf{Div}_{X/k}}$, this shows that U is open.

If we switch back to arbitrary T again, (3.4.1) implies that $D \in \mathrm{GIDiv}_{X/k}(T)$ if and only if $\varphi(T) \subseteq U$, proving the last assertion. $\square$

Remark 3.4.2. If K is any algebraically closed field extension of k, then $\mathrm{GIDiv}_{X/k}(K)$ is just the set of Cartier divisors on X_K that correspond to a Weil *prime* divisor, or in other words, the set of locally principal Weil prime divisors on X_K.

Definition 3.4.3. For a given Cartier divisor D on X, we define $\mathbf{P}_D$ and $\mathbf{Sc}_{D,Y}$ to be the scheme-theoretic intersections

$$\mathbf{P}_D = \mathbf{P}_{D/X/k} = \mathbf{L}_{D/X/k} \cap \mathbf{GIDiv}_{X/k}$$
$$\mathbf{Sc}_{D,Y} = \mathbf{Sc}_{D/(Y\to X)/k} = \widetilde{\mathbf{Sc}}_{D/(Y\to X)/k} \cap \mathbf{GIDiv}_{X/k}$$

Let $s_c^Y(D) = s_c(D)$ and $p(D)$ denote the dimensions of $\mathbf{Sc}_{D,Y}$ and $\mathbf{P}_D$, respectively.

$\mathbf{P}_D$ represents the geometrically integral divisors that are linearly equivalent to D, whereas $\mathbf{Sc}_{D,Y}$ represents those which addditionally are c-split in Y.

Since $\mathbf{P}_D$ is an open subscheme of $\mathbf{L}_D$, its dimension is $h^0(X, D) - 1$ unless it is empty. Of course, we are looking for nontrivial cases; by the following proposition, some are given by very ample divisors:

Proposition 3.4.4 (Bertini's theorem)**.** *Assume* $\dim X > 1$*. Let D be a very ample Cartier divisor on X. Then $\mathbf{P}_D$ is nonempty.*

Proof. It is enough to prove that $\mathbf{P}_D(\bar{k})$ is nonempty. But this corresponds to the set of integral divisors in the linear system $|D_{\bar{k}}|$. By Bertini's theorem, the generic member of $|D_{\bar{k}}|$ is irreducible ([Fl99, 3.4.10]) and reduced ([Fl99, 3.4.14]), hence integral. □

By proposition 3.3.1, we have

Corollary 3.4.5. *Assume* $\dim X \geq 2$*, and let D be a (very) ample divisor and D' be an arbitrary divisor on X. Then*

$$p(mD + D') = \frac{\int_X D^d}{d!} \cdot m^d + O(m^{d-1}).$$

3.5 Decomposition of divisors in branched covers

In this section, assume k is a field of *characteristic zero*. Let $f : Y \to X$ be a finite (branched) cover of normal geometrically integral projective varieties over k; assume that $\dim Y = \dim X =: d \geq 2$.

Proposition/Definition 3.5.1. Let D be a big Cartier divisor on X. Then there exists a (possibly reducible) closed subvariety $\mathbf{S}_{D,Y}$ of $\mathbf{P}_D$ such that for every field extension K/k, $\mathbf{S}_{D,Y}(K)$ is the set of all geometrically integral divisors on X_K that are linearly equivalent to D_K and split in Y_K. Denote the dimension of $\mathbf{S}_{D,Y}$ by $s^Y(D) = s(D)$. Then

$$\limsup_{m\to\infty} \frac{s(mD)}{m^d/d!} \leq \frac{1}{\deg(f)^{d-1}} \operatorname{vol}(D).$$

Proof. Since all divisors considered here are geometrically integral, we can base change to the algebraic closure of k without changing the statement. So assume that k is algebraically closed.

If Y (and hence every Y_K) is nonsingular, the notions of Weil and Cartier divisors coincide, so $\mathbf{S}_{D,Y} = \mathbf{Sc}_{D,Y}$ and the statement follows from proposition 3.3.11. Otherwise, let $\nu : Y' \to Y$ be a resolution of singularities, and let X' the pushout of $\operatorname{Spec} k(X) \leftarrow \operatorname{Spec} k(Y) \hookrightarrow Y'$.[2] The morphism $f \circ \nu : Y' \to X$ factors through X', giving a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\nu} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{\mu} & X \end{array}$$

[2]Locally, X' is glued from affine open subschemes $\operatorname{Spec}(B_i \cap k(X))$, where the $\operatorname{Spec} B_i$ form an open affine cover of Y.

with a finite cover $f' : Y' \to X'$ and a modification $\mu : X' \to X$. X' is $\mathbb{Q}$-factorial: For every Weil divisor D' on X', nD' is Cartier, where $n = \deg(f)$. To see this, we can assume D' to be a Weil prime divisor. Then there exists a Weil prime divisor E' on Y' lying over D' (i.e., $f'(E') = D'$), and $f'_*E' = [K(E') : K(D')]D'$. Since E' is Cartier and $[K(E') : K(D')]|n$, so is nD'.

Now if a prime Cartier divisor $\tilde{D} \in \operatorname{Div}(X_K)$ splits in Y_K, then its strict transform splits in Y'_K; but unfortunately, strict transforms do not preserve linear equivalence, so we have to deal with the pullback $\mu^*\tilde{D}$ instead, which equals the strict transform plus some exceptional divisor. Though $\mu^*\tilde{D}$ does not have to split in Y'_K, we can say the following: Let $D'_1, \ldots, D'_r$ be the exceptional Weil prime divisors of $X' \to X$, and let $N = \{(n_i)_i \in \{0, \ldots, n-1\}^r | \sum_{i=1}^r n_i D'_i \text{ is Cartier}\}$. Then for any $\tilde{D} \in \operatorname{Div}(X_K)$, there exists an $(n_i)_i \in N$ such that $\mu^*\tilde{D} + \sum_{i=1}^r n_i (D'_i)_K$ is the sum of the strict transform of $\tilde{D}$ and the n-fold of a Weil divisor on X'_K with support in the exceptional locus of $X'_K \to X_K$, and by what we said above, $\tilde{D}$ splits in Y_K if and only if this $\mu^*\tilde{D} + \sum_{i=1}^r n_i (D'_i)_K$ splits in Y'_K (although this divisor will not be prime any more in general, even if all n_i are zero).

Let $\mathbf{S}_{(n_i)} \subset \mathbf{P}_{D/X/k}$ denote the preimage of $\widetilde{\mathbf{Sc}}_{(\mu^*D + \sum_{i=1}^r n_i D'_i)/(Y' \to X')/k}$ under the map

$$\mathbf{P}_{D/X/k} \hookrightarrow \mathbf{L}_{D/X/k} \xrightarrow{\mu^*} \mathbf{L}_{\mu^*D/X'/k} \xrightarrow{+\sum_{i=1}^r n_i D'_i} \mathbf{L}_{(\mu^*D + \sum_{i=1}^r n_i D'_i)/X'/k},$$

and set $\mathbf{S}_{D,Y} = \bigcup_{(n_i) \in N} \mathbf{S}_{(n_i)}$. Then $\mathbf{S}_{D,Y}$ is a closed subset of $\mathbf{P}_{D/X/k}$, and if we give it the reduced closed subscheme structure, then we just proved that for any $K|k$, $\mathbf{S}_{D,Y}(K)$ gives the set of all $D' \in \mathbf{P}_D(K)$ which split in Y_K.

Furthermore, we get an upper bound on $s(D)$, namely

$$s(D) \leq \max_{(n_i) \in N} \tilde{s}_c\big(\mu^*D + \sum_{i=1}^r n_i D'_i\big).$$

Therefore, we get

$$\limsup_{m \to \infty} \frac{s(mD)}{m^d/d!} \leq \max_{(n_i) \in N} \limsup_{m \to \infty} \frac{\tilde{s}_c\big(m\mu^*D + \sum_{i=1}^r n_i D'_i\big)}{m^d/d!} \leq \frac{1}{n^{d-1}} \operatorname{vol}(\mu^*D)$$

by corollary 3.3.11. Since $\operatorname{vol}(\mu^*D) = \operatorname{vol}(D)$ by birational invariance of the volume (lemma 3.3.6), we are done. □

Having upper asymptotics for the behavior of $s(mD)$, we now show that they are also the lower asymptotics:

Lemma 3.5.2. *Let D be an ample Cartier divisor on X such that $\tilde{s}_c(D) \geq 0$. Then*

$$\liminf_{m \to \infty} \frac{s_c(mD)}{m^d/d!} \geq \frac{1}{\deg(f)^{d-1}} \operatorname{vol}(D).$$

Proof. We can assume k to be algebraically closed. Set $n := \deg(f)$. Since $\tilde{s}_c(D) \geq 0$, there is an effective Cartier divisor E_1 on Y such that $f_* E_1 \sim D$.

Define $a_m \in \mathbb{N}, b_m \in \{0, 1, \dots, n-1\}$ by $m = a_m n + b_m$. Then the divisor $E_m := a_m f^* D + b_m E_1$ has the property that $f_* E_m \sim mD$, and for $m \gg 0$, E_m is very ample. Set

$$\mathbf{U}_m = \mathbf{P}_{E_m/Y/k} \cap \mathbf{f}_*{}^{-1}(\mathbf{P}_{mD/X/k}) \subseteq \mathbf{P}_{E_m/Y/k}.$$

It is clear that $\mathbf{U}_m$ is an open subscheme of $\mathbf{P}_{E_m/Y/k}$; we claim that it is also dense. Assuming this for the moment, corollary 3.4.5 implies

$$\lim_{m \to \infty} \frac{\dim \mathbf{U}_m}{m^d/d!} = \frac{1}{n^d} \operatorname{vol}(f^* D),$$

On the other hand, by remark 3.2.2, the image of $\mathbf{U}_m$ under the finite morphism

$$\mathbf{f}_* : \mathbf{L}_{E_m/Y/k} \to \mathbf{L}_{mD/X/k}$$

lands in $\mathbf{Sc}_{mD/(Y \to X)/k}$, so $s_c(mD) \geq \dim \mathbf{U}_m$, which proves

$$\liminf_{m \to \infty} \frac{s_c(mD)}{m^d/d!} \geq \frac{1}{n^d} \operatorname{vol}_Y(f^* D) = \frac{1}{n^{d-1}} \operatorname{vol}_X(D)$$

by lemma 3.3.6.

So all that is left to show is that $\mathbf{U}_m$ is dense in $\mathbf{P}_{E_m/Y/k}$ — in other words, that $\mathbf{U}_m$ is nonempty. To do this, assume the converse, i.e. $\mathbf{f}_*(\mathbf{P}_{E_m/Y/k}) \cap \mathbf{P}_{mD/X/k} = \emptyset$. For the moment, also assume that $Y \to X$ has no proper intermediate covers $Y \to Y' \to X$. Then our assumption implies by remark 3.2.2 b) that for every $E' \in \mathbf{P}_{E_m/Y/k}(k)$, we have $f_* E' = \deg(f) \cdot f(E')$; if we construct the Galois closure $Z \to X$ of $Y \to X$ as in section 3.2.1, then $h^* E'$ is G-invariant by remark 3.2.2 c). So the proper morphism $\mathbf{h}^* : \mathbf{Div}_{Y/k} \to \mathbf{Div}_{Z/k}$ maps $\mathbf{P}_{E_m/Y/k}$ into the subscheme of G-invariant points of $\mathbf{Div}_{Z/k}$; since this subscheme is closed, it actually contains the image of $\mathbf{L}_{E_m/Y/k}$. Thus the difference of any two divisors in $|E_m|$ is $\operatorname{div}(f)$ for some $f \in K(Z)^G = K(X)$, and since the set of all those f generates $K(Y)$ (because E_m is ample), we get $K(Y) = K(X)$, hence $Y = X$ and $f = \operatorname{id}_X$, which gives a contradiction to the assumption $\mathbf{f}_*(\mathbf{P}_{E_m/Y/k}) \cap \mathbf{P}_{mD/X/k} = \emptyset$.

In the more general case, we get $\mathbf{f}''_*(\mathbf{P}_{E_m/Y/k}) \cap \mathbf{P}_{f''_* E_m/Y'/k} \neq \emptyset$ for every intermediate cover $Y \xrightarrow{f''} Y' \xrightarrow{f'} X$ such that $Y \to Y'$ has no proper intermediate covers. This implies that

$$\mathbf{V}_m = \mathbf{P}_{E_m/Y/k} \cap \bigcap_{Y \xrightarrow{f''} Y' \xrightarrow{f'} X} \mathbf{f}''^{-1}_*(\mathbf{P}_{f''_* E_m/Y'/k})$$

is dense in $\mathbf{P}_{E_m/Y/k}$, where Y' runs through the intermediate covers mentioned before; using remark 3.2.2 b) again, we can see that $\mathbf{V}_m = \mathbf{U}_m$, which finishes the proof. $\square$

Proposition 3.5.3. *Let D be an ample Cartier divisor on X such that $\tilde{s}_c(D) \geq 0$. Then*

$$\lim_{m\to\infty} \frac{s(mD)}{m^d/d!} = \lim_{m\to\infty} \frac{s_c(mD)}{m^d/d!} = \lim_{m\to\infty} \frac{\tilde{s}_c(mD)}{m^d/d!} = \frac{1}{\deg(f)^{d-1}} \operatorname{vol}(D).$$

Proof. This follows immediately from corollary 3.3.11, proposition 3.5.1, lemma 3.5.2 and the fact that $s_c(mD) \leq \min(s(mD), \tilde{s}_c(mD))$. □

Proposition 3.5.4. *Let D be an ample Cartier divisor on X with $s_c(D) \geq 0$. Then $d_c(mD) := \frac{s_c(mD)}{p(mD)}$ and $d(mD) := \frac{s(mD)}{p(mD)}$ converge as m approaches ∞, and*

$$\lim_{m\to\infty} d_c(mD) = \lim_{m\to\infty} d(mD) = \frac{1}{\deg(f)^{d-1}}.$$

Proof. This follows directly from corollary 3.4.5 and proposition 3.5.3. □

Remark 3.5.5. In the last proposition (and the ones before that), we can drop the condition $s_c(D) \geq 0$ if we rephrase our assertion. Given any ample Cartier divisor D on X, nD c-splits in Y, where $n = \deg(f)$; proposition 3.5.1 and lemma 3.5.2 prove that in this case,

$$\limsup_{m\to\infty} d(mD) = \lim_{m\to\infty} d(mnD) = \frac{1}{\deg(f)^{d-1}},$$

with analogous statements for d_c.

3.6 A Chebotarev density theorem for divisors

Assume that $Z \to X$ is a finite (branched) Galois cover with Galois group G of varieties of dimension $d \geq 2$ over a field k of characteristic zero. Let us recall the notion of a decomposition group: For a point $z \in Z$, the decomposition group of z is just the stabilizer $G_z = \{\sigma \in G | \sigma z = z\}$ of z. To define the *decomposition class* $\mathcal{C}_x$ of a point $x \in X$, choose a point $z \in Z$ lying over x and let $\mathcal{C}_x$ be G_z modulo conjugation (so $\mathcal{C}_x$ is a conjugacy class of subgroups of G instead of an actual subgroup).

Theorem 3.6.1. *Given a conjugacy class $\mathcal{C}$ of subgroups of G and an ample divisor D on X, there exist a subvariety $\mathbf{D}_D^{\mathcal{C}}$ of $\mathbf{P}_D$ representing the normal, geometrically integral divisors that are linearly equivalent to D, are unramified in $Z \to X$ and have decomposition class $\mathcal{C}$. If $H < G$ is any representative of $\mathcal{C}$, $\mathbf{D}_D^{\mathcal{C}}$ can be realized as an open subvariety of $\mathbf{S}_{D,Z/H}$. If we set $d_{\mathcal{C}}(D) := \frac{\dim \mathbf{D}_D^{\mathcal{C}}}{p(D)}$, then*

$$\limsup_{m\to\infty} d_{\mathcal{C}}(mD) = \lim_{m\to\infty} d_{\mathcal{C}}(maD) = \frac{1}{[G:\mathcal{C}]^{d-1}}$$

for some positive integer a. We can always take $a = \operatorname{ord} G$ (even $a = \operatorname{ord} \mathcal{C}$ [3]); if some effective divisor linearly equivalent to D is c-split in Z, we can take $a = 1$.

Proof. The proof basically comes down to the following claim:

$$\mathbf{D}_D^{\mathcal{C}} \stackrel{!}{=} \mathbf{S}_{D,Z/H} \setminus \left(\bigcup_{H' \lneq H} \mathbf{S}_{D,Z/H'} \cup \{\text{ramified locus}\} \right)$$

The ramified locus contains finitely many closed points, hence does not influence the dimension. The comparison of the asymptotic behavior of all $s^{Z/H'}(mD)$ then gives the assertion, once the claim is proven.

To show the claim, fix $Y = Z/H$ and let f, g, h be the quotient morphisms $Y \to X$, $Z \to X$, $Z \to Y$, as usual. Take any Weil prime divisor $D' \in X$ unramified in Z, let F' be a prime divisor lying above D', and let $E' = h(F')$. Since D' is unramified, we have $[k(F') : k(D')] = \operatorname{ord} G_{F'}$ and $[k(F') : k(E')] = \operatorname{ord} H_{F'} = \operatorname{ord}(G_{F'} \cap H)$, so $g_* F' = \operatorname{ord} G_{F'} D'$ and $h_* F' = \operatorname{ord}(G_{F'} \cap H) E'$. This implies $f_* E' = \frac{\operatorname{ord} G_{F'}}{\operatorname{ord}(G_{F'} \cap H)} D'$, where $\frac{\operatorname{ord} G_{F'}}{\operatorname{ord}(G_{F'} \cap H)} = 1 \Leftrightarrow G_{F'} < H$. Therefore, D' is split in Y if and only some representantive of $\mathcal{C}_{D'}$ is a subgroup of H. If we use the same argument for all $Y' = Z/H'$ with $H' < H$, we get that an unramified Weil prime divisor D' is split in $Y = Z/H$, but not in any $Y' = Z/H'$ with $H' \lneq H$ if and only if H is a representative of the decomposition class $\mathcal{C}_{D'}$. This proves the claim. □

Remark 3.6.2. The only reason why we restricted ourselves to characteristic zero in this section is the fact that we use resolution of singularities in the proof. Therefore, these results also hold for surfaces and threefolds in chararacteristic p (using [Ab98]).

[3] Here we set $\operatorname{ord} \mathcal{C} := \operatorname{ord} H$ for any representative H of $\mathcal{C}$; similarly, we set $[G : \mathcal{C}] := [G : H]$.

Chapter 4

Positive characteristics

4.1 Decomposition of divisors in étale covers

The title of this chapter might be misleading, since the results in this section are valid in any characteristic. The only difference to the results in the last section is that we have to add another condition on our morphism. Therefore, in the following, let $f : Y \to X$ be a finite *étale* cover of *geometrically normal* and geometrically integral projective varieties over a field k, and assume $\dim Y = \dim X =: d \geq 2$.

Remark 4.1.1. a) The condition that Y is geometrically normal can be omitted since it follows automatically from the fact that X is geometrically normal and f is étale ([Mi80, I.3.17]).

b) Instead of demanding that f is étale, we could use the seemingly weaker condition that f is unramified – an unramified cover $f : Y \to X$ is automatically étale because X is normal ([Mi80, I.3.20]).

c) Since f is unramified, $k(Y)|k(X)$ has to be separable.

The big improvement compared to the situation in section 3.5 is that the notion of split and c-split divisors coincide:

Proposition 4.1.2. *Let D be a geometrically normal and geometrically integral divisor on X, assume D splits in Y. Then D is c-split in Y.*

Proof. By assumption, there exists a Weil divisor E on Y such that f_*E is the Weil divisor corresponding to D. We claim that E is locally principal. In fact, since $f|_E : E \to D$ is finite and birational and D is geometrically normal and geometrically integral, $f|_E$ is an isomorphism, hence $f^*D \to D$ has a section $D \xrightarrow{\cong} E \subset f^*D$. But $f^*D \to D$ is a base change of $f : Y \to X$ and therefore étale (and separated); by [Mi80, I.3.12] this implies that E is a connected component of f^*D. Since f^*D is locally principal, so is E. $\square$

Proposition/Definition 4.1.3. There exists an open subscheme $\mathbf{GNDiv}_{X/k}$ of $\mathbf{Div}_{X/k}$ representing the functor $\mathrm{GNDiv}_{X/k}$ defined by

$$\mathrm{GNDiv}_{X/k}(T) = \left\{ D \in \mathrm{Div}_{X/k}(T) \,\middle|\, \begin{array}{c} D_t \text{ geometrically normal and} \\ \text{geometrically integral } \forall\, t \in T \end{array} \right\}.$$

For a given Cartier divisor D on X, we define $\mathbf{P}_D^{\mathbf{gn}}$ and $\mathbf{S}_{D,Y}^{\mathbf{gn}}$ to be the scheme-theoretic intersections

$$\mathbf{P}_D^{\mathbf{gn}} = \mathbf{L}_D \cap \mathbf{GNDiv}_{X/k}$$
$$\mathbf{S}_{D,Y}^{\mathbf{gn}} = \widetilde{\mathbf{Sc}}_{D,Y} \cap \mathbf{GNDiv}_{X/k}$$

For an arbitrary field extension $K|k$, $\mathbf{P}_D^{\mathbf{gn}}(K)$ represents the set of all geometrically normal and geometrically integral divisors on X_K that are linearly equivalent to D_K, and $\mathbf{S}_{D,Y}^{\mathbf{gn}}(K)$ represents the subset of divisors in $\mathbf{P}_D^{\mathbf{gn}}(K)$ which additionally split in Y_K.

Let $s_Y^{gn}(D) = s^{gn}(D)$ and $p^{gn}(D)$ denote the dimensions of $\mathbf{S}_{D,Y}^{\mathbf{gn}}$ and $\mathbf{P}_D^{\mathbf{gn}}$, respectively. If D is very ample, then $p^{gn}(D) = h^0(X, D) - 1 \geq 0$.

Proof. The proof of the existence of $\mathbf{GNDiv}_{X/k}$ is basically the same as the proof of the existence of $\mathbf{GIDiv}_{X/k}$ in proposition 3.4.1, it just additionally uses [EGAIV-3, 12.2.4 (iv)]. The representation statements follow mostly from the definitions; only for $\mathbf{S}_{D,Y}^{\mathbf{gn}}$, we have to use proposition 4.1.2. As for the nonemptyness of $\mathbf{P}_D^{\mathbf{gn}}$, the proof of proposition 3.4.4 applies; the cited proposition [Fl99, 3.4.14] also includes this result. □

Now we again arrive at a Chebotarev density result. For this, assume that $Z \to X$ is an étale Galois cover with Galois group G of varieties of dimension $d \geq 2$ over a field k of arbitrary characteristic.

Theorem 4.1.4. *Given a conjugacy class $\mathcal{C}$ of subgroups of G and an ample divisor D on X, there exist a subvariety $\mathbf{D}_D^{\mathbf{gn},\mathcal{C}}$ of $\mathbf{P}_D^{\mathbf{gn}}$ representing the geometrically normal, geometrically integral divisors that are linearly equivalent to D and have decomposition class $\mathcal{C}$. If $H < G$ is any representative of $\mathcal{C}$, $\mathbf{D}_D^{\mathbf{gn},\mathcal{C}}$ can be realized as an open subvariety of $\mathbf{S}_{D,Z/H}^{\mathbf{gn}}$. If we set $d_{\mathcal{C}}^{gn}(D) := \frac{\dim \mathbf{D}_D^{\mathbf{gn},\mathcal{C}}}{p^{gn}(D)}$, then*

$$\limsup_{m\to\infty} d_{\mathcal{C}}^{gn}(mD) = \lim_{m\to\infty} d_{\mathcal{C}}^{gn}(maD) = \frac{1}{[G:\mathcal{C}]^{d-1}}$$

for some positive integer a. We can take $a = \operatorname{ord}\mathcal{C}$; if some divisor linearly equivalent to D is split in Z, we can take $a = 1$.

Proof. Using $\mathbf{D}_D^{\mathbf{gn},\mathcal{C}} = \mathbf{S}_{D,Z/H}^{\mathbf{gn}} \setminus \bigcup_{H' \lneq H} \mathbf{S}_{D,Z/H'}^{\mathbf{gn}}$ (which is proved in the proof of theorem 3.6.1) and $p^{gn}(mD) = \frac{m^d}{d!}\operatorname{vol}(D) + O(m^{d-1})$, it is enough to prove that for a finite étale cover $f : Y \to X$ of geometrically normal and geometrically integral k-varieties of dimension $d \geq 2$,

$$\limsup_{m\to\infty} \frac{s^{gn}(mD)}{m^d/d!} = \lim_{m\to\infty} \frac{s^{gn}(mnD)}{(mn)^d/d!} = \frac{1}{n^{d-1}} \operatorname{vol}(D),$$

where $s^{gn} = s^{gn}_Y$ and $n = \deg(f)$.

The upper inequality $\limsup_{m\to\infty} \frac{s^{gn}(mD)}{m^d/d!} \leq \frac{1}{n^{d-1}} \operatorname{vol}(D)$ follows from corollary 3.3.11 and the fact that $\mathbf{S}^{\mathbf{gn}}_{D,Y} \subset \widetilde{\mathbf{Sc}}_{D,Y}$. So it remains to show that for a very ample divisor with $\tilde{s}_c(D) \geq 0$, we have

$$\liminf_{m\to\infty} \frac{s^{gn}(mD)}{m^d/d!} \geq \frac{1}{n^{d-1}} \operatorname{vol}(D).$$

But this follows immediately from an analog of lemma 3.5.2 for s^{gn}, which can be proved similarly. $\square$

4.2 Finite fields

The case of finite fields deserves some special attention, since here we can actually *count* divisors. We will therefore give another definition of density in this case, relying on the number of divisors in a linear series instead of its dimension. We have to be careful in translating our previous results, though: Even though we know $\mathbf{P}_D$ to be an open dense subvariety of the projective space $\mathbf{L}_D$ if D is very ample, its number of rational points can theoretically lie anywhere between 0 and $\#\mathbf{L}_D(k)$. Therefore, we will not be able to deduce these new results as simple corollaries from what we have done so far.

So let us assume that $k = \mathbb{F}_q$ is a finite field, and that $f : Y \to X$ is a cover fulfilling the conditions in the beginning of the last section.

Definition 4.2.1. For a given Cartier divisor D on X, let $p_\#(D)$ denote the number of geometrically normal and geometrically integral divisors in the linear system $|D|$.

Since $p_\#(D) \leq \#|D| = \frac{q^{h^0(X,D)}-1}{q-1}$, we know that $\log_q p_\#(D) < h^0(X, D)$; the following proposition asserts that for mD with increasing m, this inequality gives an asympotic:

Proposition 4.2.2. *Let D be an ample divisor on X. Then*

$$\lim_{m\to\infty} \frac{\log_q p_\#(mD)}{m^d/d!} = \operatorname{vol}(D).$$

Proof. We will show $\liminf_{m\to\infty} \frac{p_{\#}(mD)}{\#H^0(X,mD)} \geq C$ for some positive constant C; from this the assertion follows easily using the definition of the volume.

We can assume D to be very ample. Let $X \hookrightarrow \mathbb{P}^N$ be the projective embedding corresponding to the very ample linear system $|D|$. Then for $m \geq h^0(X, \mathcal{O}_X) - 1$, the map

$$\phi_m : S_m := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, mD)$$

is surjective ([Po04, 2.1]); thus

$$\frac{p_{\#}(mD)}{\#H^0(X,mD)} = \frac{\#(\mathcal{P} \cap S_m)}{\#S_m},$$

where $\mathcal{P}$ denotes the set of all $f \in S_{\mathrm{homog}} := \bigcup_{m=0}^{\infty} S_m$ such that the scheme-theoretic intersection $H_f \cap X$ of the hypersurface H_f of $\mathbb{P}^N$ defined by f with X is geometrically normal and geometrically integral.

We will now give a list of conditions on $f \in S_{\mathrm{homog}}$ which are sufficient for f to be in $\mathcal{P}$, from this we will then deduce the asserted inequality using the Bertini theorem over finite fields in [Po04]. Before doing this, let us fix some notation: Let $\bar{k}$ denote the algebraic closure of k, and let $\mathrm{Reg}(X)$ and $\mathrm{Sing}(X)$ denote the regular resp. singular locus of X (since X is geometrically normal, $\mathrm{Reg}(X)$ is smooth and $\mathrm{codim}(\mathrm{Sing}(X), X) \geq 2$).

Using Serre's criterion, we have $f \in \mathcal{P} \Leftrightarrow (H_f \cap X)_{\bar{k}} = (H_f)_{\bar{k}} \cap X_{\bar{k}}$ is connected and fulfills R_1 and S_2. Now we can split this up into several sufficient conditions:

- In order for $(H_f \cap X)_{\bar{k}}$ to be connected, it suffices by Grothendieck's connectedness theorem ([Fl99, 3.1.1]) that H_f is geometrically integral (here we use that X is geometrically integral and of dimension ≥ 2, hence $X_{\bar{k}}$ is connected in dimension 1 by [Fl99, 3.1.3(3)]). Thus, set $\mathcal{Q}_1$ to be the set of all $f \in S_{\mathrm{homog}}$ such that H_f is *not* geometrically integral.

- In order to consider Serre's condition S_2, define

  $$Z_r(Y) = \{y \in Y \,|\, \mathrm{codepth}\, \mathcal{O}_{Y,y} := \dim \mathcal{O}_{Y,y} - \mathrm{depth}\, \mathcal{O}_{Y,y} > r\}$$

  for any Noetherian scheme Y. Then Y fulfills S_2 if and only if

  $$\mathrm{codim}(Z_r(Y), Y) > r + 2 \text{ for all } r \geq 0 \quad \text{([EGAIV-2, 5.7.4]).}$$

  Since $X_{\bar{k}}$ is normal, we know that $\mathrm{codim}(Z_r(X_{\bar{k}}), X_{\bar{k}}) > r + 2 \ \forall\ r \geq 0$. On the other hand, if $x \in X_{\bar{k}}$ lies in $(H_f)_{\bar{k}}$ ($f \neq 0$), then f maps to a regular element of $\mathfrak{m}_x$, so $\mathrm{codepth}\, \mathcal{O}_{Y,y} = \mathrm{codepth}\, \mathcal{O}_{Y,y}/(f)$ by [EGAIV-1, 0.16.4.10(i)]; thus $Z_r((X \cap H_f)_{\bar{k}} = Z_r(X_{\bar{k}}) \cap (X \cap H_f)_{\bar{k}}$. So in order for f to fulfill $\mathrm{codim}(Z_r((X \cap H_f)_{\bar{k}}), (X \cap H_f)_{\bar{k}}) > r+2 \ \forall\, r \geq 0$, it will be sufficient that $(H_f)_{\bar{k}}$ intersects all irreducible components of all $Z_r(X_{\bar{k}})$ properly, i.e. $(H_f)_{\bar{k}}$

does not contain any of the irreducible components of any $Z_r(X_{\bar{k}})$. There are only finitely such irreducible components, since $Z_r(X_{\bar{k}})$ is empty for $r \geq d$.

Set Z to be the finite reduced subscheme of X consisting of all closed points[1] which are an irreducible component of either one of the $Z_r(X_{\bar{k}})$ or of $\mathrm{Sing}(X_{\bar{k}})$, and set $\mathcal{Q}_2$ to be the set of all $f \in S_{\text{homog}}$ such that $(H_f)_{\bar{k}}$ contains at least one of the positive dimensional irreducible components of either one of the $Z_r(X_{\bar{k}})$ or of $\mathrm{Sing}(X_{\bar{k}})$.

- We now turn towards the R_1 property. If $f \notin \mathcal{Q}_2$ and $H_f \cap Z = \emptyset$, then $\mathrm{codim}(\mathrm{Sing}(X_{\bar{k}}) \cap (H_f)_{\bar{k}}, (X \cap H_f)_{\bar{k}}) \geq 2$; thus $(X \cap H_f)_{\bar{k}}$ has R_1 if $U \cap H_f$ has, where $U = \mathrm{Reg}(X)$. So it is sufficient that $U \cap H_f$ is smooth of dimension $d-1$. If $T \subset H^0(Z, \mathcal{O}_Z)$ is the (nonempty) set of all sections which don't vanish at any point of Z, we set

$$\mathcal{P}' := \{f \in S_{\text{homog}} : H_f \cap U \text{ is smooth of dimension } d-1, \text{ and } f|_Z \in T\}.$$

Putting all these pieces of information together, we have

$$\mathcal{P} \supset \mathcal{P}' - (\mathcal{Q}_1 \cup \mathcal{Q}_2).$$

Now using Poonen's Bertini theorem for finite fields ([Po04, 1.2]), we have

$$\lim_{m\to\infty} \frac{\#(\mathcal{P}' \cap S_m)}{\#S_m} = \frac{\#T}{\#H^0(Z, \mathcal{O}_Z)} \zeta_U(d+1)^{-1} =: C > 0,$$

where ζ_U is the zeta function of U, and $\lim_{m\to\infty} \frac{\#(\mathcal{Q}_1 \cap S_m)}{\#S_m} = 0$ by [Po04, 2.7]; so we will be done as soon as we show that $\lim_{m\to\infty} \frac{\#(\mathcal{Q}_2 \cap S_m)}{\#S_m} = 0$. In fact, all we have to show is that given any irreducible subvariety W of X of positive dimension and $\mathcal{Q} := \{f \in S_{\text{homog}} | W \subset H_f\}$, then $\lim_{m\to\infty} \frac{\#(\mathcal{Q} \cap S_m)}{\#S_m} = 0$. But $\frac{\#S_m}{\#(\mathcal{Q} \cap S_m)} = h^0(W, O_W(m)) \xrightarrow{m\to\infty} \infty$ by Riemann-Roch, so we are done. $\square$

Definition 4.2.3. For a given Cartier divisor D on X, let $s^Y_\#(D) = s_\#(D)$ denote the number of geometrically normal and geometrically integral divisors in the linear system $|D|$ which split in Y. We set

$$d_\#(D) = \frac{\log_q s_\#(D)}{\log_q p_\#(D)}.$$

Proposition 4.2.4. *Let D be an ample divisor on X. Then*

$$\limsup_{m\to\infty} d_\#(mD) = \lim_{m\to\infty} d_\#(mnD) = \frac{1}{n^{d-1}},$$

where $n = \deg(f)$.

[1] to be more precise, images of such points under $X_{\bar{k}} \to X$

Proof. The fact that $\liminf_{m\to\infty} \frac{\log_q s_\#(mnD)}{m^d n^d/d!} \geq \frac{\mathrm{vol}(D)}{n^{d-1}}$ follows similarly to the proof for proposition 3.5.2 using proposition 4.2.2, so we can concentrate on the proof of the upper bound.

By the proof of theorem 4.1.2, we know that $s_\#(mD) \leq \#\widetilde{\mathbf{Sc}}_{mD,Y}(k)$. Using the proof of lemma 3.2.4, we get that

$$s_\#(mD) \leq \sum_{f_*[E]_{num}=[mD]_{num}} \#\,\mathbf{Div}_{Y/k}^{[E]_{num}}(k). \tag{4.2.1}$$

In order to give a bound on the number of summands occuring in this sum, let us remind ourselves of the notation introduced before proposition 3.3.9: H_D is the affine subspace of $N^1(Y)_\mathbb{R}$ consisting of all elements which map to $[D]_{num}$ under $f_* : N^1(Y)_\mathbb{R} \to N^1(X)_\mathbb{R}$. The intersection of $N^1(Y)$ with H_D is a full lattice in H_D; in particular, it defines a natural volume form on H_D such that the fundamental lattice has volume 1. The intersection of the pseudoeffective cone $\overline{\mathrm{Eff}}(Y)$ with H_D is compact, hence has finite volume V; by standard combinatorial arguments, one can deduce that

$$\#\{\eta \in \overline{\mathrm{Eff}}(Y) \cap N^1(Y) | f_*\eta = [mD]_{num}\} = Vm^l + O(m^{l-1}), \tag{4.2.2}$$

where l is the dimension of H_D.

On the other hand, a k-rational point of $\mathbf{Div}_{Y/k}^{[E]_{num}}$ maps to a k-rational point of $\mathbf{Pic}_{Y/k}^{[E]_{num}}$ under the Abel map; considering the fibers of this map, we get

$$\#\,\mathbf{Div}_{Y/k}^{[E]_{num}}(k) \leq \#\,\mathbf{Pic}_{Y/k}^{\tau}(k) \cdot q^{\max_{E'\equiv E}\{h^0(X,E')\}}. \tag{4.2.3}$$

Using (4.2.2) and (4.2.3) in (4.2.1), we get

$$\log_q s_\#(mD) \leq \max\{h^0(Y,E) | E \in \mathrm{Div}(Y), f_*E \equiv mD\} + l \log_q m + C$$

for some constant $C > 0$. Now using proposition 3.3.9, this implies

$$\limsup_{m\to\infty} \frac{\log_q s_\#(mD)}{m^d/d!} \leq \frac{\mathrm{vol}(D)}{\deg(f)^{d-1}}. \qquad \square$$

Theorem 4.2.5. *Let $Z \to X$ be an étale Galois cover with Galois group G of varieties of dimension $d \geq 2$ over a finite field k; let $\mathcal{C}$ be a conjugacy class $\mathcal{C}$ of subgroups of G and D an ample divisor on X. Set*

$$d_\#^\mathcal{C}(D) := \frac{\log_q \#\mathbf{D}_D^{\mathbf{gn},\mathcal{C}}(k)}{\log_q p_\#(D)}.$$

Then

$$\limsup_{m\to\infty} d_\#^\mathcal{C}(mD) = \lim_{m\to\infty} d_\#^\mathcal{C}(maD) = \frac{1}{[G:\mathcal{C}]^{d-1}}$$

for some positive integer a. We can take $a = \mathrm{ord}\,\mathcal{C}$; if some divisor linearly equivalent to D is split in Z, we can take $a = 1$.

Proof. The proof follows easily from proposition 4.2.4 using the fact that

$$s_{\#}^{Z/H}(D) - \sum_{H' \subsetneq H} s_{\#}^{Z/H'}(D) \leq \#\mathbf{D}_D^{\mathbf{gn},\mathcal{C}}(k) \leq s_{\#}^{Z/H}(D),$$

where H is a representative of $\mathcal{C}$; and these inequalities are evident from the proof of theorem 3.6.1. □

Chapter 5

Applications

In this chapter, we will consider some applications of the density theorems. All varieties in this chapter will be projective, geometrically normal and geometrically integral over a field k and have fixed dimension $d \geq 2$; all morphisms will be finite (branched) covers, in case $\operatorname{char} k > 0$ we additionally assume the morphisms to be étale.

5.1 A generalization of a theorem of Bauer

We now turn our attention towards the comparison of *split* and *completely split* divisors. Bauer seems to have been the first one to realize that Galois covers can be identified as those where the notions of split and completely split agree; he specified Galois extensions number fields in the following way:

Theorem 5.1.1 (Bauer, [Ne92, 13.8]). *Let $L|K$ be an extension of number fields. If every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ that is unramified and split in L is completely split in L, then $L|K$ is Galois.*

Here the notions of split and completely split are the usual ones in number theory; they are the equivalents of our notions in the case of number fields.

We will prove a similar result in our situation. Let $f : Y \to X$ be a cover of varieties over k as described above. Before we go on, let us remember the notation of section 3.2.1: Since we are either in characteristic zero, or $f : Y \to X$ is étale, $k(Y)|k(X)$ is separable. Let Z be the normalization of X in the Galois closure of $k(Y)|k(X)$, and $H < G = \operatorname{Aut}_X(Z)$ be finite groups acting on Z such that $X = Z/G$, $Y = Z/H$.

Proposition 5.1.2. *Let $K|k$ be an arbitrary field extension. For a prime Cartier divisor D on X_K, the following are equivalent:*

a) D splits completely in Y_K,

b) D splits completely in Z_K,

c) D is unramified and splits in Z_K.

Proof. b) $\Leftrightarrow$ c) and b) $\Rightarrow$ a) are trivial, for a) $\Rightarrow$ b) assume that D splits completely in Y_K. Let F be a Weil prime divisor on Z_K with $g(F) = D$. Then $(\sigma F)_{\sigma \in G}$ runs through all Weil prime divisors on Z_K above D ([La02, VII.2.1]), and $h(\sigma F)$ runs through all Weil prime divisors on Y_K above D. By our assumption on D, we have $n = \deg(f)$ different Weil prime divisors on Y_K lying above D, so $h(\sigma F) = h(\tau\sigma F) \Leftrightarrow \tau \in H$. If $G_{\sigma F}$ denotes the decomposition group of σF, this shows $\sigma G_F \sigma^{-1} = G_{\sigma F} \subseteq H \; \forall \; \sigma \in G$, hence $G_F \subseteq \bigcap_{\sigma \in G} \sigma^{-1} H \sigma$. But the last group is normal in G and contained in H, so trivial by construction of Z. Thus G_F is trivial, i.e. D splits completely in Z_K. □

Remark 5.1.3. Proposition 5.1.2 is not true any more if we replace complete splitting by complete c-splitting.

As an easy *affine* example of this, consider the cone $Z = \operatorname{Spec} R$ with $R = \mathbb{C}[z_1, z_2, z_3]/(z_1^2 + z_2^2 + z_3^2)$. The group $G = S_3$ acts on Z by permutation of the z_i. Let H be the subgroup of G generated by the permutation (12), and set $X = Z/G$ and $Y = Z/H$. Then $f : Y \to X$ is a finite cover with Galois closure $g : Z \to X$.

Both Y and X are isomorphic to $\mathbb{A}^2$: If we set $y_1 = z_1 + z_2, y_2 = z_1 z_2$, then $Y = \operatorname{Spec} R^H$ with $R^H = \mathbb{C}[y_1, y_2, z_3]/(y_1^2 - 2y_2 + z_3^2) = \mathbb{C}[y_1, z_3]$; if we set x_1, x_2, x_3 to be the elementary symmetric polynomials in z_1, z_2, z_3, then $X = \operatorname{Spec} R^G$ with $R^G = \mathbb{C}[x_1, x_2, x_3]/(x_1^2 - 2x_2) = \mathbb{C}[x_1, x_3]$. In particular, X and Y are smooth, and the notions of Weil and Cartier divisors coincide on them.

Z, on the other hand, contains Weil divisors which are not Cartier. As an example, take the Weil prime divisor F: $z_1 = z_2 + iz_3 = 0$ on Z, which is a ruling of the cone and not a Cartier divisor by [Ha77, II.6.5.2]. Set $g_* F = D$. Since the decomposition group of F is trivial, D is a (geometrically integral) Cartier divisor on X which is completely c-split in Y, but not completely c-split in Z.

Proposition/Definition 5.1.4. Assume $\operatorname{char} k = 0$. Let D be an ample Cartier divisor on X. Then there exists a (possibly reducible) open dense subvariety $\mathbf{T}_{D,Y}$ of $\mathbf{S}_{D,Z}$ such that for every field extension K/k, $\mathbf{T}_{D,Y}(K)$ is the set of all geometrically integral divisors on X_K that are linearly equivalent to D_K and split completely in Y_K. We denote its dimension by $t(D)$.

There exists a positive integer a such that $\tilde{d}(maD) := \frac{t(maD)}{p(maD)}$ converges as m approaches ∞, and

$$\limsup_{m \to \infty} \tilde{d}(mD) = \lim_{m \to \infty} \tilde{d}(maD) = \frac{1}{\operatorname{ord}(G)^{d-1}},$$

where G is the Galois group of the Galois closure of $k(Y)|k(X)$.

Proof. Using the notation from before, proposition 5.1.2 shows that we can take $\mathbf{T}_{D,Y} = \mathbf{T}_{D,Z}$ to be $\mathbf{S}_{D,Z} \backslash \mathbf{g}_*\big(\bigcup_{\sigma \in G, \sigma \neq \mathrm{id}} \{z \in \mathbf{Div}_{Z/k} \,|\, \sigma z = z\}\big)$. Since G acts faithfully on Z, $\mathbf{T}_{D,Y}$ is (open and) dense in $\mathbf{S}_{D,Z}$. The statement about the limit follows from remark 3.5.5; it turns out that we can take $a = \mathrm{ord}(G)$. □

Theorem 5.1.5. *Assume that there exists a very ample Cartier divisor D on X such that we have $\mathbf{S}_{mD,Y} \subseteq \mathbf{T}_{mD,Y}$ (or just $s(mD) \leq t(mD)$) for $m \gg 0$. Then $f : Y \to X$ is a Galois cover.*

Proof. By possibly choosing a positive multiple of D, we can assume that $s_c(D) \geq 0$. Then the condition implies

$$\frac{1}{n^{d-1}} = \lim_{m \to \infty} d(mD) \leq \limsup_{m \to \infty} \tilde{d}(mD) = \frac{1}{(\mathrm{ord}\, G)^{d-1}}, \tag{5.1.1}$$

so $[k(Z) : k(X)] = \mathrm{ord}(G) \leq n = [k(Y) : k(X)]$. But $k(Y) \subseteq k(Z)$, so $k(Y) = k(Z)$, and $f : Z = Y \to X$ is a Galois cover. □

Remark 5.1.6. In arbitrary characteristic, assume moreover $f : Y \to X$ to be étale. Then $\mathbf{T}^{\mathbf{gn}}_{D,Y} = \mathbf{T}^{\mathbf{gn}}_{D,Z} := \mathbf{S}^{\mathbf{gn}}_{D,Z}$ represents the geometrically normal and geometrically integral divisors which are linearly equivalent to D and split completely in Y (the description is simpler since there is no ramification), and we get similar versions of proposition 5.1.4 and theorem 5.1.5.

Theorem 5.1.7. *Let $f : Y \to X$ be a finite cover of projective, geometrically normal, geometrically integral varieties over a field k of dimension at least 2; if k has positive characteristic, assume furthermore that f is étale. Then the following are equivalent:*

a) f is a Galois cover.

b) Every unramified Weil prime divisor on $X_{\bar{k}}$ that splits in $Y_{\bar{k}}$ is completely split.

c) There exists an ample Cartier divisor D on X such that for all $m \gg 0$, every geometrically integral Cartier divisor on $X \in |mD|$ and splits in Y is completely split in Y.

Proof. Assume first $\mathrm{char}\, k = 0$. The only nontrivial implication is c) $\Rightarrow$ a), i.e. that $\mathbf{S}_{mD,Y}(k) \subseteq \mathbf{T}_{mD,Y}(k)$ for $m \gg 0$ already implies that $f : Y \to X$ is Galois. But by the proof of proposition 3.5.2, $\mathbf{S}_{mnD,Y}$ (where $n = \deg(f)$) contains an open dense subset of $\mathbf{f}_*\mathbf{P}_{mf^*D}$, which is unirational (thus has a dense subset of k-rational points), so we get

$$\dim \mathbf{T}_{mnD,Y} \geq \dim \mathbf{P}_{mf^*D},$$

which for $m \to \infty$ leads to the desired equation (5.1.1). Then everything follows similar to the proof of theorem 5.1.5.

In positive characteristic, use $\mathbf{T}^{\mathbf{gn}}_{mnD,Y}$ instead of $\mathbf{T}_{mnD,Y}$. The same argument as above works as long as k is infinite; if k is finite, we get $s^Z_\#(mnD) \geq \frac{1}{n} p_\#(mf^*D)$, so

$$\begin{aligned}\frac{n^d}{(\operatorname{ord} G)^{n-1}} \operatorname{vol}(D) &= \lim_{m\to\infty} \frac{\log_q s^Z_\#(mnD)}{m^d/d!} \\ &\geq \lim_{m\to\infty} \frac{\log_q p_\#(mf^*D)}{m^d/d!} = \operatorname{vol}(f^*D) = n \operatorname{vol}(D),\end{aligned}$$

which again leads to equation (5.1.1). □

For finite fields, the proof of the last theorem shows that it can be decided by counting whether f is Galois:

Theorem 5.1.8. *In the situation of Theorem 5.1.7, let k be a finite field. In addition to the notation used in section 4.2, let $t_\#(D)$ denote the number of geometrically normal, geometrically integral divisors linearly equivalent to a given divisor D which additionally are completely split in Y. Then the following are equivalent:*

a) f is an (étale) Galois cover.

b) There exists an ample Cartier divisor D on X such that

$$\limsup_{m\to\infty} \frac{\log t_\#(mD)}{\log p_\#(mD)} = \frac{1}{\deg(f)^{n-1}}.$$

Indeed, we can generalize our first theorem to criteria using subvarieties of higher codimension, too. Let us first remark that the notion of splitting and complete splitting make sense for any irreducible closed subvariety V of X: We say that V is split in Y if there exists a irreducible closed subvariety W of Y such that $f(W) = V$ and $k(W) = k(V)$; V is said to be completely split in Y if there are $n = \deg(f)$ different such W.

Theorem 5.1.9. *Let $f : Y \to X$ be a finite cover of projective, geometrically normal, geometrically integral varieties over a field k of dimension $d \geq 2$; if k has positive characteristic, assume furthermore that f is étale. Then the following are equivalent:*

a) f is a Galois cover.

b) There exists a positive integer $r < d$ such that every closed subvariety V of X of codimension r that is unramified and splits in Y is completely split.

Proof. We will give the proof in the case $\operatorname{char} k = 0$, the proof for $\operatorname{char} k > 0$ is similar. a) $\Rightarrow$ b) is immediate, so we have to show that if $Y \to X$ is not Galois and $r < d$ is any positive integer, there exists a subvariety of codimension r which is unramified, split, but not completely split. If $r = 1$, we are done by theorem 5.1.7. Otherwise, construct the Galois cover $g : Z \to X$ of $Y \to X$ as in section 3.2.1 and take $\mathcal{C}$ to be the congugacy class of G. By theorem 3.6.1, we know there exists a geometrically integral divisor D which is unramified in Z and has decomposition class $\mathcal{C}$. Take $X_1 = D$ and let Y_1 and Z_1 be the preimages in Y and Z, respectively. Then $Z_1 \to X_1$ is a Galois cover of geometrically integral varieties over k of dimension $d - 1 > r - 1 \geq 1$. We can proceed to get closed subvarieties $X \supset X_1 \supset \ldots \supset X_{r-1}$, such that $\dim X_i = d - i$ and $Z_i = X_i \times_X Z \to X_i$ is a Galois cover of geometrically integral varieties over k with Galois group G for every $i = 1, \ldots, r-1$. Inside X_{r-1}, we can find a geometrically integral divisor X_r of decomposition class $\mathcal{C}(H)$, again using theorem 3.6.1. X_r is a closed subvariety of X of codimension r, and it is split in $Y_{r-1} = X_{r-1} \times_X Y$ (and therefore in Y) by the proof of 3.6.1, but is not split in Z_{r-1} and thus not completely split in Y_{r-1} for the same reasons. This finishes the proof. $\square$

5.2 Bauerian covers

Additionally to the general conditions mentioned at the beginning of the chapter, fix a variety X and an ample divisor D on X. Define the $M(D)$ to be the monoid consisting of all effective divisors on X which are linearly equivalent to some integer multiple of D.

We have an inclusion of monoids $M(D) = \bigcup_{m=0}^{\infty} |mD| \subset \operatorname{Div}(X) \subseteq Z^1(X)$.

Definition 5.2.1. For any cover $f : Y \to X$, set

$$S(Y) = S(Y, D) = f_* Z^1(Y) \cap M(D)$$

We call $S(Y)$ the *Kronecker monoid* for the cover $Y \to X$.

If f is étale, we can recover all $\mathbf{S}^{\mathbf{gn}}_{mD,Y}(k)$ from $S(Y)$, since

$$S(Y) \cap \mathbf{GNDiv}_{X/k}(k) = \bigcup_{m=0}^{\infty} \mathbf{S}^{\mathbf{gn}}_{mD,Y}(k).$$

Similarly, in the case $\operatorname{char} k = 0$, we can recover all $\mathbf{S}_{mD,Y}(k)$ from $S(Y)$ by intersecting $S(Y)$ with $\mathbf{GIDiv}_{X/k}(k)$.

Theorem 5.2.2. *Let $Y \to X$ be an finite étale Galois cover, $Y' \to X$ be an arbitrary finite étale cover. Then*

$$S(Y') \subseteq S(Y) \quad \Leftrightarrow \quad Y' \to X \text{ factors through } Y \to X.$$

Proof. The only implication which is not obvious is that $S(Y') \subseteq S(Y)$ implies that $Y' \to X$ factors through $Y \to X$. To see this, take a Galois extension $L|k(X)$ which contains both $k(Y)$ and $k(Y')$ and such that k is algebraically closed in L; let Z be the normalization of X in L. Setting $G = \mathrm{Gal}(L|k(X))$, G acts on Z as mentioned in section 3.2.1, and with $H = \mathrm{Gal}(L|k(Y)) \triangleleft G$, $H' = \mathrm{Gal}(L|k(Y')) < G$, we have $Z/G \cong X$, $Z/H \cong Y$ and $Z/H' \cong Y'$. Let $\mathcal{C}$ be the congujacy class of H'. Using theorem 4.1.4 for infinite fields and theorem 4.2.5 for finite fields, we get that for some $m \gg 0$, there exist a geometrically normal, geometrically integral divisor D' linearly equivalent to mD with decomposition class $\mathcal{C}$. This implies $D' \in \mathbf{S}^{\mathbf{gn}}_{mD,Y'}(k)$, so by our assumption, we get $D' \in \mathbf{S}^{\mathbf{gn}}_{mD,Y}(k)$. As was explained in the proof of theorem 3.6.1, D' being split in Y implies that some representative of $\mathcal{C}$ is a subgroup of H. Since H is normal in G, we get $H' < H$, thus $Y' \cong Z/H' \to X$ factors through $Y \cong Z/H \to X$. □

Corollary 5.2.3. *A Galois cover $Y \to X$ is completely described by its Kronecker monoid.*

Corollary 5.2.4. *The natural transformation*

$$\{\textit{open normal subgroups of } \pi_1^{\acute{e}t}(X)\} \to \{\textit{submonoids of } M(D)\}$$
$$H \mapsto S(Y_H)$$

is fully faithful. Here $Y_H \to X$ denotes the étale Galois cover of X corresponding to the normal subgroup $H \triangleleft \pi_1^{\acute{e}t}(X)$.

Definition 5.2.5. A cover $Y \to X$ is called *Bauerian* if for any other cover $Y' \to X$,

$$S(Y') \subseteq S(Y) \Rightarrow Y' \to X \text{ factors through } Y \to X.$$

Corollary 5.2.6. *Galois covers are Bauerian.*

More or less all of these statements are analogs of well-known applications of the original Chebotarev density theorem. Of course, the Chebotarev density theorem in the original version or in the one of Serre has some even more far-reaching consequences: Apart from their connections to class field theory, these versions are also used in the proofs of the theorems of Neukirch, Uchida ([Uc77]) and Pop ([Pop94]) which state that if K and L are two global fields or fields finitely generated over $\mathbb{Q}$ with isomorphic Galois groups, then K and L are isomorphic.

It might be worthwhile to examine whether similar results can be shown using the Chebotarev-like theorems in this thesis; on the other hand, since the results mentioned above can hardly be regarded as pure applications of Chebotarev's density theorem (the density theorem is only used for one small step), many more obstacles should be expected, if at all a generalization to our case is possible.

Bibliography

[Ab98] ABHYANKAR, S. S.: *Resolution of singularities of embedded algebraic surfaces.* Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[Ch25] CHEBOTAREV, N.: *Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören*, Math. Ann. 95 (1925), 191–228.

[De01] DEBARRE, O.: *Higher-Dimensional Algebraic Geometry.* Universitext. Springer-Verlag, New York, 2001.

[EGAII] GROTHENDIECK, A. *Éléments de géométrie algébrique II.* Étude globale élémentaire de quelques classes de morphismes. Inst. Hautes Études Sci. Publ. Math., No. 8, 1961

[EGAIV-1] GROTHENDIECK, A. *Éléments de géométrie algébrique IV.* Étude locale des schémas et des morphismes de schémas I. Inst. Hautes Études Sci. Publ. Math., No. 20, 1964

[EGAIV-2] GROTHENDIECK, A. *Éléments de géométrie algébrique IV.* Étude locale des schémas et des morphismes de schémas II. Inst. Hautes Études Sci. Publ. Math., No. 24, 1965

[EGAIV-3] GROTHENDIECK, A. *Éléments de géométrie algébrique IV.* Étude locale des schémas et des morphismes de schémas III. Inst. Hautes Études Sci. Publ. Math., No. 28, 1966

[EGAIV-4] GROTHENDIECK, A. *Éléments de géométrie algébrique IV.* Étude locale des schémas et des morphismes de schémas IV. Inst. Hautes Études Sci. Publ. Math., No. 32, 1967

[FAG05] FANTECHI, B. ET AL.: *Fundamental algebraic geometry.* Grothendieck's FGA explained. Mathematical Surveys and Monographs, 123. American Mathematical Society, Providence, RI, 2005.

[Fl99] Flenner, H. et al.: *Joins and Intersections*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[Fuj94] Fujita, T.: *Approximating Zariski decomposition of big line bundles*. Kodai Math. J. 17 (1994), no. 1, 1–3.

[Fu98] Fulton, W.: *Intersection Theory*, Second edition. Springer-Verlag, Berlin, 1998.

[Ha77] Hartshorne, R.: *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Kl98] Klingen, N.: *Arithmetical similarities*. Prime decomposition and finite group theory. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998

[La02] Lang, S.: *Algebra*, Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.

[La04a] Lazarsfeld, R.: *Positivity in Algebraic Geometry I*. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004.

[La04b] Lazarsfeld, R.: *Positivity in Algebraic Geometry II*. Positivity for vector bundles, and multiplier ideals. Springer-Verlag, Berlin, 1992.

[Mi80] Milne, J.: *Étale Cohomology*, Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980.

[Ne92] Neukirch, J.: *Algebraische Zahlentheorie*. Springer-Verlag, Berlin, 2004.

[Po04] Poonen, B.: *Bertini Theorems over finite fields*, Ann. of Math. (2) 160 (2004), no. 3, 1099–1127.

[Pop94] Pop, F.: *On Grothendieck's conjecture of birational anabelian geometry*, Ann. of Math. (2) 138 (1994), no. 1, 145–182.

[Se65] Serre, J.-P. *Zeta and L functions*. Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963) 82–92. Harper & Row, New York, 1965.

[SGA6] Berthelot, P. et al.: Séminaire de Géométrie Algébrique du Bois Marie (1966-67) - Théorie des intersections et théorème de Riemann-Roch (SGA 6). Lecture notes in mathematics 225. Springer-Verlag, Berlin-New York, 1971.

[Ta07] Takagi, S.: *Fujita's approximation theorem in positive characteristics*, J. Math. Kyoto Univ. 47 (2007), no. 1, 179–202.

[Uc77] Uchida, K.: *Isomorphisms of Galois groups of algebraic function fields*, Ann. of Math. (2) 106 (1977), no. 1, 589–598.