Seminar

# $p$-adic Weil II (after Kedlaya)

Sommersemester 2013

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**Ort**: HS 4 **Zeit**: 11 Uhr c.t. **Beginn**: 18.04.2013

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## Contents



## 1 Aim

Let $q = p^r$ for a fixed prime $p$. Let $k$ be a perfect field of characteristic $p$ containing $\mathbb{F}_q$, let $\mathfrak{o}$ be a finite totally ramified extension of the ring of Witt vectors $W(k)$, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and let $K$ denote the fraction field of $\mathfrak{o}$. One assumes that $\mathfrak{o}$ admits an automorphism $\sigma_K$ lifting the $q$-power Frobenius map. We fix once and for all an embedding $\iota : \overline{K} \hookrightarrow \mathbb{C}$.

By a **variety** we understand a separated scheme of finite type over $\mathbb{F}_q$. Berthelot defines a $p$-adic cohomology theory, **rigid cohomology** (and a version with compact support), which seems reasonable (as we shall see) for any variety, and which generalizes crystalline cohomology in the smooth proper case and Monsky-Washnitzer cohomology in the smooth affine case. The role of coefficients is played by so-called **overconvergent $F$-isocrystals**, which are sheaves on certain analytic spaces. Here $F$ corresponds to a Frobenius, so that there exists a theory of weights, with respect to the embedding $\iota$.

The aim of the seminar is to understand the proof of the following $p$-adic analogue of Weil II over a point:

**Theorem 1.1.** Let $X$ be a variety over $\mathbb{F}_q$ and $\mathcal{E}$ be a $\iota$-realizable overconvergent $F$-isocrystal on $X$.

1. If $\mathcal{E}$ is $\iota$-mixed of weight $\leq w$ then $H^i_{c,\mathrm{rig}}(X/K, \mathcal{E})$ is $\iota$-mixed of weight $\leq w + i$ for each $i$.
2. If $X$ is smooth and $\mathcal{E}$ is $\iota$-mixed of weight $\geq w$, then $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ is $\iota$-mixed of weight $\geq w + i$ for each $i$.

We will closely adhere to Kedlaya's papers [4] and [5]. We give an outline of the techniques and the proof.

## 2 Dagger algebras and $(\sigma, \nabla)$-modules

Rigid cohomology is effectively computed in the smooth affine case using the theory developed by Monsky and Washnitzer, which involves modules over certain dagger algebras. Almost every proof of Kedlaya is reduced to an explicit computation using these modules.

For $\rho > 1$ one defines the rings

$$T_{n,\rho} := \{\sum_I a_I x^I |\ a_I \in K,\ \lim_{\sum I \to \infty} |a_I| \rho^{\sum I} = 0\},$$

where $I = (i_1, \ldots, i_n)$ are non-negative integers, $x^I = x_1^{i_1} \cdots x_n^{i_n}$ and $\sum I = i_1 + \ldots + i_n$. This is an affinoid algebra and may be considered as power series converging on a disc of radius strictly greater than 1, i.e., they "overconverge". Set

$$K\langle x_1, \ldots, x_n\rangle^\dagger := W_n := \bigcup_{\rho > 1} T_{n,\rho},$$

considered as a subset of the usual Tate-algebra. A **dagger algebra** $A$ is an algebra isomorphic to a quotient of $W_n$ for some $n$. One can endow any such algebra with the so-called fringe topology and a lift of a Frobenius $\sigma$.

Let $\Omega^i_{A/K}$ denote the usual (continuous) differential module for $i \geq 1$, and let $d : A \to \Omega^1_{A/K}$ be the canonical derivation. A $(\sigma, \nabla)$**-module** is a finite locally free $A$-module $M$ that comes equipped with a $\sigma$-linear map $F : M \to M$ and an integrable connection $\nabla : M \to M \otimes_R \Omega^1_{A/K}$.

## 3 Robba-rings

Crew was able to compute cohomology with compact support (at least in the case of a curve) using the theory of Robba rings, under the assumption of the $p$-adic local monodromy conjecture. In 2001, different proofs were given of this conjecture, which are essential in establishing the properties of rigid cohomology.

Here, the **Robba ring** $\mathcal{R}_K$ is the ring of bidirectional power series that converge for $t \in \overline{K}$ with $0 < \eta < |t| < 1$, i.e.

$$\mathcal{R}_K := \{ \sum_{n=-\infty}^{\infty} c_n t^n |\ c_n \in K,\ \exists r = r(\eta) > 0 : \lim_{n \to \pm\infty} (v_p(c_n) + rn) = \infty\}.$$

In a more general vein, Kedlaya defines Robba rings $\mathcal{R}_A$ over a dagger algebra $A$, which are fundamental when dealing with higher dimensional varieties.

Similarly as in the previous section, one may define differential modules and $(\sigma, \nabla)$-modules over these Robba rings.

## 4 Rigid cohomology

The construction of rigid cohomology (at least starting from the smooth case) is quite natural. The method uses formal schemes (which allows lifting from characteristic $p$ to characteristic 0) and rigid analytic spaces.

Let $P$ be a $\mathfrak{o}$-formal scheme. The **generic fiber** $\widetilde{P}$ of $P$ is defined as the set of closed formal subschemes of $P$ which are integral, finite and flat over $\mathfrak{o}$. It comes with a specialization map $\mathrm{sp} : \widetilde{P} \to P$. For instance, if $P$ is affine, then $P = \mathrm{Spf}(R)$ for some complete topologically finitely generated $\mathfrak{o}$-algebra $R$, and $\widetilde{P} = \mathrm{MaxSpec}(R \otimes_{\mathfrak{o}} R)$, so that $\widetilde{P}$ has the structure of a rigid analytic space.

If $Y \hookrightarrow P$ is any subvariety of the special fiber of $P$, the **tube** $]Y[$ of $Y$ is defined as $\mathrm{sp}^{-1}(Y)$, which, as can be shown, also has the structure of an analytic space. If $Y' \hookrightarrow Y \hookrightarrow P$ are embeddings of subvarieties, there is also the notion of a **strict neighborhood** $V$ of $]Y'[$ in $]Y[$, which we won't explicate here.

We now fix a variety $X$ with compactification $\overline{X}$. Suppose that there exists a closed immersion (as locally ringed spaces) of $\overline{X}$ into a $\mathfrak{o}$-formal scheme $P$ which is smooth in a neighborhood of $X$ (the general definition is by a glueing construction). Using this, one defines an **overconvergent $F$-isocrystal** $\mathcal{E}$ on $X$ to be a finite locally free $\mathcal{O}_V$-module on some strict neighborhood $V$ of $]X[$, equipped with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_V$ and an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ that is subject to a certain convergence condition.

The **rigid cohomology** of $\mathcal{E}$ is then defined as

$$H^i_{\mathrm{rig}}(X/K, \mathcal{E}) = H^i(]X[, j^\dagger(\mathcal{E} \otimes \Omega^\cdot_V)),$$

where $j^\dagger$ is the limit over all inclusions $j : ]X[ \hookrightarrow V$ for the strict neighborhood associated to $\mathcal{E}$. For instance, if $X = \overline{X}$ is smooth and proper and one takes the trivial isocrystal, then

$$H^i_{\mathrm{rig}}(X/K) = H^i(]X[, \Omega^\cdot) = H^i_{\mathrm{crys}}(X/K).$$

The important property here is that if $X$ is smooth affine, the limit $A = \varinjlim \Gamma(V, \mathcal{O}_V)$ over all embeddings $j : ]X[ \to V$ into a strict neighborhood is a dagger-algebra, which in turn induces an equivalence of categories between the category of overconvergent $F$-isocrystals and the category of $(\sigma, \nabla)$-modules over $A$. This equivalence extends to cohomology, so that $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ may be computed as the cohomology of the complex

$$\ldots \longrightarrow M \otimes \Omega^i_{A/K} \xrightarrow{\nabla_i} M \otimes \Omega^{i+1}_{A/K} \longrightarrow \ldots,$$

where $M$ is the $A$-module corresponding to $\mathcal{E}$, which is nothing but the definition of (classical) Monsky-Washnitzer cohomology in the smooth affine case.

Similarly, due to Crew (in the case of a curve), there exist certain embeddings $A \hookrightarrow \mathcal{R}_x$ for each closed point $x \in \overline{X}$, where $\mathcal{R}_x = \mathcal{R}_{K'}$ is a copy of the Robba ring over some finite unramified extension $K'/K$, and one defines

$$\rho : A \hookrightarrow A_{\mathrm{loc}} := \bigoplus_{x \in \overline{X} \setminus X} \mathcal{R}_x,$$

$A_{\mathrm{qu}} := A_{\mathrm{loc}}/A$ and $\Omega^1_{A,\mathrm{qu}} := \Omega^1_{A/K} \otimes_A A_{\mathrm{qu}}$. One can show that the complex

$$\ldots \longrightarrow M \otimes \Omega^i_{A,\mathrm{qu}} \xrightarrow{\nabla_i} M \otimes \Omega^{i+1}_{A,\mathrm{qu}} \longrightarrow \ldots$$

then computes $H^i_{c,\mathrm{rig}}(X/K, \mathcal{E})$, if $M$ again corresponds to $\mathcal{E}$. We note that there is of course a generalization of this construction (again due to Kedlaya) to higher dimensional affine varieties.

# 5 Properties of rigid cohomology

Let $\mathcal{E}$ be an overconvergent $F$-isocrystal on a variety $X$. After establishing a generalization of the monodromy conjecture for Robba rings over dagger algebras (which we will assume as a black box in the seminar), Kedlaya proves the following results, using explicit computations in the smooth affine case:

1. The cohomology groups $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ and $H^i_{\mathrm{c,rig}}(X/K, \mathcal{E})$ are finite dimensional $K$-vector spaces for all $i$,

2. One has a natural perfect pairing (the **Poincaré duality**)

$$H^i_{\mathrm{rig}}(X/K,\mathcal{E}) \otimes_K H^i_{\mathrm{c,rig}}(X/K,\mathcal{E}^\vee) \longrightarrow K,$$

   where $\mathcal{E}^\vee$ is the dual $F$-isocrystal.

He also establishes a Künneth-formula which we won't use here.

# 6 Strategy for the proof of Weil II

We now briefly outline the proof of Theorem 1.1. The sections here refer to the sections in [5].

Section 3 shows a degeneration statement for a morphism $f : K\langle s\rangle^\dagger \to K\langle s,x\rangle^\dagger$, i.e., a map that is relative of dimension 1. That is, if $M$ is a $(\sigma,\nabla)$-module over $K\langle s,x\rangle^\dagger$, under some conditions on the cohomology of $M$ (depending on an integer $m$), then $R^1 f_\# M^{(\vee)}$ is free of rank $m$ over $K\langle s\rangle^\dagger$ (Proposition 3.4.3), where $\# \in \{!,*\}$ and $M^{(\vee)}$ means either $M$ or the dual $M^\vee$.

Section 4 is concerned with the construction of a $p$-adic Fourier transform. One introduces the the ring $\mathcal{D}^\dagger$ of overconvergent differential operators on $K\langle s\rangle^\dagger$ and shows that $(\sigma,\nabla)$-modules over $K\langle s\rangle^\dagger$ are actually $\mathcal{D}^\dagger$-modules. For any such $M$, the **Fourier transform** $\widehat{M}$ of $M$ is defined as $\widehat{M} := \mathcal{D}^\dagger \otimes_\rho M$, where $\rho$ is a certain automorphism of $\mathcal{D}^\dagger$.

In section 4.3 Kedlaya defines the **Dwork isocrystal** $\mathcal{L}$ on $K\langle s\rangle^\dagger$. Further, for $f \in A$, where $A$ is any dagger algebra, one has a map $K\langle s\rangle^\dagger \to A$, $x \mapsto f$, so that one defines $\mathcal{L}_f := f^*\mathcal{L}$. If $M$ is as before and $f : K\langle s\rangle^\dagger \to K\langle s,x\rangle^\dagger$, $g : K\langle x\rangle^\dagger \to K\langle s,x\rangle^\dagger$ are the canonical embeddings, one defines

$$N = g^*M \otimes_{K\langle s,x\rangle^\dagger} \mathcal{L}_{sx}.$$

Denote by $\nabla_s$ the component of the connection on $N$ mapping into the rank one submodule of $\Omega^1_{K\langle s,x\rangle^\dagger}$ generated by $ds$. Hence, one may define the **geometric Fourier transform** as $\widehat{M}_{\mathrm{geom}} := \mathrm{coker}\nabla_x$. One exhibits a canonical isomorphism $\widehat{M} \to \widehat{M}_{\mathrm{geom}}$ of $\mathcal{D}^\dagger$-modules (Proposition 4.3.1), which is used to show that $\widehat{M}$ is absolutely irreducible as a $(\sigma,\nabla)$-module, if certain conditions on the dimension the the $H^i$'s of $M^{(\vee)} \otimes \mathcal{L}_?$ hold (Proposition 4.3.2).

Section 4.4 shows an analogue of the Grothendieck-Ogg-Shafarevich formula (Theorem 4.4.1). It implies a result on the dimension of the $H^i$'s of $M^{(\vee)} \otimes \mathcal{L}_?$ (Proposition 4.4.5). The upshot is (using Proposition 4.3.2) that $M$ is absolutely irreducible if and only if its Fourier transform is absolutely irreducible.

Section 5 shows the Lefschetz trace formula

$$\prod_{x\in X} \det(1 - F_x t^{\deg(x)}, \mathcal{E}_x)^{-1} = \prod_i \det(1 - Ft, H^i_{\mathrm{c,rig}}(X/K,\mathcal{E})^{(-1)^{i+1}},$$

using the Dwork operator $\psi$ on $M$ (we will assume this result).

Section 6.1 recalls the theory of weights for an endomorphism of a finite dimensional vector space over $K$, with respect to $\iota$. For instance, an operator $T$ is called **$\iota$-pure of weight** $w$ if for each eigenvalue $\alpha$ of $T$ one has $|\iota(\alpha)| = q^{(w/2)}$. Then, if $\mathcal{E}$ is an overconvergent $F$-isocrystal, one says that $\mathcal{E}$ is **$\iota$-real** if the coefficients of the characteristic polynomial of the linear transformation $F_x$ on $\mathcal{E}_x$ for $x \in X(\mathbb{F}_q)$ map into $\mathbb{R}$. Further, $\mathcal{E}$ is called **$\iota$-realizable** if it is a direct summand of an $\iota$-real overconvergent $F$-isocrystal.

Section 6.2 recalls the monodromy formalism developed by Deligne which gives the absolute value of the eigenvalues for certain central elements.

As a consequence, section 6.3 shows that any irreducible $\iota$-realizable overconvergent $F$-isocrystal on a curve $X$ is $\iota$-pure of some weight (Theorem 6.3.4).

Section 6.5 then proves, if $\mathcal{E}$ is an $F$-isocrystal on $\mathbb{A}^1$ that is $\iota$-pure of weight $w$, that $H^1(M)$ and $H^1_c(M)$ are $\iota$-pure of weight $w+1$, using Proposition 3.4.3, the results on the Fourier transform (Proposition 4.3.2) and the purity result of Theorem 6.3.4.

By degenerating this result, one obtains (using Katz' so-called "weight drop lemma", Lemma 6.4.3), if $\mathcal{E}$ is $\iota$-mixed of weight $\geq w$, that $H^1_{\mathrm{rig}}(\mathbb{A}^1/K, \mathcal{E})$ is $\iota$-mixed of weight $\geq w+1$ (Theorem 6.5.3).

Finally, the proof of Theorem 6.6.2 in section 6.6 shows the statement of Theorem 1.1. By Poincaré duality and excision, one is reduced to the case $X = \mathbb{A}^n$ and the proof of b) in loc.cit., which in turn is reduced via a projection $f : \mathbb{A}^n \to \mathbb{A}^{n-1}$ to the case of $\mathbb{A}^1$, where one may finally invoke Theorem 6.5.3.

## 7 List of talks

The first three talks introduce the basic notions of the seminar. Some aspects of these talks are intended to be of an expository nature, since unfortunately we do not have the time to study rigid cohomology thoroughly (see however [6] for a complete reference). Talks four to six gather the main results of [4] in the smooth affine case, which is (due to a reduction argument) mostly what we need. The last five talks stick closely to [5] and try to fit in most of the proofs there.

We note that we reference the arXiv versions of Kedlaya's papers since they appear to be more "verbose".

1. **Formal and rigid geometry, dagger algebras**: rigid analytic spaces ([4] 2.1; also [7] and [2]) and the connection with formal schemes (e.g. [7]), strict neighborhoods ([1] section 2 and [4] Definition 4.2.2), dagger algebras, fringe algebras & fringe topology ([4] 2.2 and 2.3).

2. **Robba rings and $(\sigma, \nabla)$-modules**: Robba rings over $K$ and $A$, module of differentials, algebras of MW-type, Frobenius lifts, $(\sigma, \nabla)$-modules ([4] 2.5 - 3.3).

3. **Rigid cohomology**: $F$-isocrystals ([1] section 4 and [4] Definition 4.2.3), rigid cohomology ([1] section 2 and 4.2.4 and [4] Definition 4.3.1), comparison with MW-cohomology ([1] 2.1), cohomology with compact support ([4] Definition 4.3.4), comparison with Kedlaya's construction for affine $n$-space ([4] Proposition 8.1.8; compare also [5] 2.6 and [3] 8.1.3).

4. **Finite dimensionality of $H^i_{\mathbf{rig}}$**: near isomorphisms ([4] 6.1), Crews' construction of the map $\rho$ ([3] 7.2-7.3 and [4] 6.2), [4] Proposition 6.2.10, only cite [4] 6.3, finite dimensionality in the absolute case ([4] Theorem 6.4.2; with necessary references from [4] section 5; we assume Theorem 5.1.3).

5. **Push-forwards and base-change**: definition of push-forwards, base change ([4] 7.2, 7.3), [4] Theorem 7.3.3 about higher direct images, flat base change ([4] Proposition 7.5.2).

6. **Poincaré duality**: Poincaré pairing ([4] 8.2), duality for affine $n$-space ([4] Proposition 8.5.3), relative Poincaré ([4] Proposition 8.6.1).

7. **Degeneration in families**: rank statements for $R^1 f_{\#} M^{(\vee)}$, where $f : K\langle s\rangle^\dagger \to K\langle s, x\rangle^\dagger$ ([5] p.21-24).

8. **$p$-adic Fourier transform**: the ring $\mathcal{D}^\dagger$, $\mathcal{D}^\dagger$-modules, the Fourier transform $\widehat{M}$, the geometric Fourier transform $\widehat{M}_{\text{geom}}$ ([5] p.24-29).

9. **$\widehat{M} \cong \widehat{M}_{\text{geom}}$ and Euler characteristic**: [5] Proposition 4.3.1, Euler characteristic formula [5], Proposition 4.4.5 ([5] p.30-p.35 middle).

10. **Weights and global monodromy**: pure and mixed weights, $\iota$-realizability ([5] 6.1), purity result for rank 1 isocrystals ([5] 6.1.2 - 6.1.4), global monodromy formalism ([5] 6.2.1, 6.2.2, 6.2.3, sketch only).

11. **Proof of the main theorem**: properties of determinantal weights, local monodromy, Weil II on the affine line, Rigid Weil II ([5] p.44-49).

## References


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[2] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

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[4] K. S. Kedlaya. Finiteness of rigid cohomology with coefficients. http://arxiv.org/abs/math/0208027, August 2002. ArXiv Mathematics e-prints, 0208027.

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